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Received: 8 February 2021 | Accepted: 24 February 2022

DOI: 10.1002/mana.202100074

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Nonzero positive solutions of fractional Laplacian systems with functional terms

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Funding information
GNAMPA 2020/21 Project "Metodi topologici per problemi al contorno associati a certe classi di equazioni alle derivate parziali", Grant/Award Number: Project Metodi topologici per problemi al conto

Abstract

We study the existence of nonzero positive solutions of a class of systems of differential equations driven by fractional powers of the Laplacian. Our approach is based on the notion of fixed point index, and allows us to deal with nonlocal functional weights and functional boundary conditions. We present two examples to shed light on the type of functionals and growth conditions that can be considered with our approach.

KEYWORDS
cone, fixed point index, fractional Laplacian, functional boundary condition, nonlocal functional weights, positive solution

MSC (2020)
Primary 35R11; Secondary 35B09, 35A16, 47H10

1 | INTRODUCTION AND PRELIMINARIES

The main aim of this paper is to establish some existence and nonexistence results for "functional" Dirichlet problems driven by fractional powers of the classical Laplace operator. More precisely, if $m \geq 1$ is a fixed natural number, we shall be concerned with Dirichlet problems of the following form:

$$\begin{cases} (-\Delta)^{s_i} u_i = \lambda_i f_i(x, \mathbf{u}, \mathcal{P}_i[\mathbf{u}]) & \text{in } \Omega \qquad (i = 1, \dots, m), \\ u_i \equiv \eta_i \zeta_i(x) \mathcal{B}_i[\mathbf{u}], & \text{in } \mathbb{R}^n \setminus \Omega \qquad (i = 1, \dots, m), \\ \mathbf{u} \gneqq 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a fixed open set, $\mathbf{u} = (u_1, \dots, u_m)$ and, for $i = 1, \dots, m$,

(1) f_i is a real-valued function defined on $\Omega \times \mathbb{R}^m \times \mathbb{R}$;
(2) λ_i, η_i are nonnegative parameters;

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(3) ζ_i is a sufficiently regular, real-valued function defined on $\mathbb{R}^n$;
(4) $\mathcal{P}_i$, $\mathcal{B}_i$ are suitable functionals to be defined later.

Moreover, $s_1, \dots, s_m \in (0,1)$ and $(-\Delta)^{s_i}$ denotes the standard fractional Laplace operator of order s_i, which is the nonlocal operator defined as

$$(-\Delta)^{s_i} v(x) = c_{n,s_i} \cdot \text{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+2s_i}} \, \mathrm{d}y.$$

Here, $c_{n,s_i} > 0$ is the "normalization" constant defined as

$$c_{n,s_i} := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(y_1)}{|y|^{n+2s_i}} \, \mathrm{d}y \right)^{-1}.$$

Notice that, in addition to the fractional differential operators, in system (1.1) other nonlocal terms occur, both in the differential equations (having the role of *nonlocal functional weights*) and in the boundary conditions (BCs). In particular, we are interested in the existence/nonexistence of *positive* solutions of (1.1), and our approach is based on the classical notion of fixed point index in cones. We work in the Banach space of the bounded continuous $\mathbb{R}^m$-valued functions defined in $\mathbb{R}^n$, namely,

$$\mathbb{X} := \left\{ u \in C(\mathbb{R}^n; \mathbb{R}^m) : \sup_{\mathbb{R}^n} |u_i| < \infty \ \text{ for all } i = 1, \dots, m \right\},$$

endowed with the supremum norm; accordingly, since we are interested in nonzero positive solutions, we look for solutions of (1.1) lying in the cone

$$P := \{ \mathbf{u} \in \mathbb{X} : \ u_i \geq 0 \text{ on } \mathbb{R}^n \text{ for every } i = 1, \dots, m \}.$$

In view of these facts, it is natural to assume that the real-valued operators $\mathcal{P}_i$, $\mathcal{B}_i$ are defined on $\mathbb{X}$ (for all $i = 1, \dots, m$).

As it is by now well known, equations involving the fractional Laplacian arise in several applications; because of this, they have been extensively studied in the last decades by many authors, we provide as a reference the comprehensive survey [13]. Among others, let us mention here the equations, driven by the fractional Laplacian, which have additional nonlocal terms and are often referred to as *Kirchhoff-type equations*. For instance, Kirchhoff-type equations on bounded domains have been recently studied in [8, 11, 15, 28], while *systems* on bounded domains are investigated, for example, in [12]. Moreover, Kirchhoff-type equations on the whole of $\mathbb{R}^n$ have been studied in [3, 6, 7, 25]. As regards *concrete* "real-word" applications, the kind of problems that we are able to deal with seems to be of interest in, for example, biological models: Indeed, on one hand, equations with functional terms on the right-hand side commonly appear in models about cell adhesion (see, e.g., [18, 22]); on the other hand, the fractional Laplacian is also used to model superdiffusive cells (see, e.g., [14, 23]).

In the above cited papers, variational methods are frequently used to prove the existence/multiplicity of solutions. To the best of our knowledge, not many papers have been devoted to equations driven by the fractional Laplacian from the point of view of topological methods. Let us mention here, for instance, the recent papers by Alves, de Lima, and Nóbrega [1, 2]: In these papers, the authors obtained Rabinowitz-type global bifurcation results of positive solutions of a parametric fractional Laplacian equation in $\mathbb{R}^n$ using the Leray–Schauder degree. On the other hand, due to the presence of the nonlocal functional weights, system (1.1) can be viewed as a *nonlinear fractional Kirchhoff-type problem*, even if in a slightly different direction than the one proposed in [12, 15].

As already pointed out, in this paper, we adopt a topological approach based on the classical notion of fixed point index (see, e.g., [17]) to prove our main existence result, namely, Theorem 3.3 below; moreover, we prove a nonexistence result via an elementary argument. In some sense, our existence result stems from a pioneering work by Amman [4, 5] and follows a line recently pursued by the authors in the study of elliptic PDEs [9, 19–21]. We point out that our approach permits to consider (possibly nonlinear) functional BCs: For example, in Section 4, we will discuss the solvability of the following boundary value problem (BVP)

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$$\begin{cases} (-\Delta)^{\frac{1}{4}} u_1 = \lambda_1 (1-u_1) \int_{B_1} e^{u_2}\,\mathrm{d}x & \text{in } B_1, \\ (-\Delta)^{\frac{3}{4}} u_2 = \lambda_2 u_2 \cdot \operatorname{osc}_{B_1}(u_1) & \text{in } B_1, \\ u_1\big|_{\mathbb{R}^2\setminus B_1} = \eta_1 \cdot u_1(0) u_2(0), \\ u_2\big|_{\mathbb{R}^2\setminus B_1} = \eta_2 \cdot \limsup_{|x|\to\infty} u_1(x), \end{cases}$$

where B_1 denotes the Euclidean ball in $\mathbb{R}^2$ centered at 0 with radius 1, and by $\operatorname{osc}_{B_1}(\phi)$, we mean the oscillation of the function ϕ on B_1.

We now briefly describe the structure of our paper. In the first part, we perform a preliminary study of the fractional differential operators, which occur in (1.1): In Section 2, we collect some properties and estimates of the solutions of the Dirichlet problem for $(-\Delta)^s$, $s \in (0,1)$, which allow to define a *Green operator*, denoted by $\mathcal{G}_s$, from $L^\infty(\Omega)$ to $C^{0,s}(\mathbb{R}^n)$. These properties are probably known to the experts in the field, nevertheless we include them for the sake of completeness. We refer the interested reader to the already quoted survey [13] for a detailed and self-contained introduction to the fractional Laplacian. We also discuss the positivity and compactness of the Green operator $\mathcal{G}_s$, thought of as an operator from $L^\infty(\Omega)$ into itself, as well as spectral properties of $\mathcal{G}_s$. Roughly speaking, these estimates yield the a priori bounds needed to compute the fixed point index in suitable cones of nonnegative functions. We point out that a challenging feature of our investigation is the choice of the appropriate functional spaces to which the solutions belong. This is discussed in detail in Section 2. In Section 3, we prove our main results, while the last Section 4 contains a couple of examples illustrating both our existence and nonexistence result.

2 | PRELIMINARIES AND AUXILIARY RESULTS

In order to keep the paper as self-contained as possible, we collect in this section some definitions and results, which shall be exploited in the sequel.

2.1 | The $(-\Delta)^s$-green operator

Here, we introduce the so-called $(-\Delta)^s$-Green operator and we establish some of its basic properties. Throughout what follows, we take for fixed all the notation listed below.

(1) $\Omega \subseteq \mathbb{R}^n$ is a (nonempty) open set with smooth boundary and $s \in (0,1)$.
(2) $H^s(\mathbb{R}^n)$ is the usual fractional Sobolev space of order s, that is,

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y < \infty \right\}.$$

(3) If $U \subseteq \mathbb{R}^n$ is any open set and $\alpha \in (0,1)$ is fixed, $C^{0,\alpha}(\overline{U})$ is the set of the functions $u : \overline{U} \to \mathbb{R}$, which are Hölder-continuous of exponent α up to $\overline{U}$, that is,

$$C^{0,\alpha}(\overline{U}) := \left\{ u \in C(\overline{U}) : [u]_{\alpha,U} := \sup_{x\neq y\in U} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\}.$$

In particular, if $u \in C^{0,\alpha}(\overline{U})$, we set

$$\|u\|_{C^{0,\alpha}(\overline{U})} := \sup_U |u| + [u]_{\alpha,U}.$$

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(4) $C_b(\mathbb{R}^n)$ is the Banach space of the continuous functions on $\mathbb{R}^n$, which are *globally bounded on* $\mathbb{R}^n$, that is,

$$C_b(\mathbb{R}^n) = C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

To introduce the $(-\Delta)^s$-Green operator, our starting point is the following notable result due to Ros–Oton and Serra [24].

Theorem 2.1 [24, Proposition 1.1]. *Let $f \in L^\infty(\Omega)$ be fixed. Then, there exists a unique (weak) solution $u_f \in H^s(\mathbb{R}^n) \cap C^{0,s}(\mathbb{R}^n)$ of the Dirichlet problem*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.1}$$

This means, precisely, that $u_f \equiv 0$ pointwise in $\mathbb{R}^n \setminus \Omega$ and

$$\frac{c_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_f(x) - u_f(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} \, dx \, dy = \int_\Omega f\varphi \, dx \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

Furthermore, there exists a constant $C = C(\Omega, s) > 0$ such that

$$\|u_f\|_{C^{0,s}(\mathbb{R}^n)} \le C \, \|f\|_{L^\infty(\Omega)}. \tag{2.2}$$

Thanks to Theorem 2.1, the following definition is well posed.

Definition 2.2. We define the $(-\Delta)^s$*-Green operator* (relative to Ω) as

$$\mathcal{G}_s : L^\infty(\Omega) \to C^{0,s}(\mathbb{R}^n), \qquad \mathcal{G}_s(f) := u_f, \tag{2.3}$$

where u_f is the unique solution of (2.1) (according to Theorem 2.1).

With Definition 2.2 at hand, we now proceed by proving some "topological" properties of $\mathcal{G}_s$. We begin with a couple of *continuity/compactness* results.

Proposition 2.3. *The operator $\mathcal{G}_s$ is continuous from $L^\infty(\Omega)$ to $C^{0,s}(\mathbb{R}^n)$.*

Proof. On account of (2.2), for every $f \in L^\infty(\Omega)$ we have

$$\|\mathcal{G}_s(f)\|_{C^{0,s}(\mathbb{R}^n)} = \|u_f\|_{C^{0,s}(\mathbb{R}^n)} \le C \, \|f\|_{L^\infty(\Omega)},$$

where $C > 0$ is a constant only depending on Ω and s. Therefore, since $\mathcal{G}_s$ is linear, we get that $\mathcal{G}_s$ is continuous. □

Proposition 2.4. *Let $\{f_k\}_{k=1}^\infty$ be a* bounded *sequence in $L^\infty(\Omega)$. Then, there exists $v_0 \in C(\mathbb{R}^n)$ such that $v_0 \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ and (up to a subsequence)*

$$\lim_{k\to\infty} \mathcal{G}_s(f_k) = v_0 \qquad \textit{uniformly in } \mathbb{R}^n. \tag{2.4}$$

In particular, $\mathcal{G}_s$ is compact from $L^\infty(\Omega)$ into $L^\infty(\mathbb{R}^n)$.

Proof. First of all, since $\{f_k\}_{k=1}^\infty$ is bounded in $L^\infty(\Omega)$, it follows from (2.2) that the family $\{\mathcal{G}_s(f_k)\}_{k=1}^\infty \subseteq C^{0,s}(\mathbb{R}^n)$ is equi-continuous and equi-bounded; as a consequence, since $\overline{\Omega}$ is compact, by Arzelà–Ascoli's theorem, there exists some function $g \in C(\overline{\Omega})$ such that (up to a subsequence)

$$\lim_{k\to\infty} \mathcal{G}_s(f_k) = g \qquad \text{uniformly on } \overline{\Omega}. \tag{2.5}$$

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In particular, since $\mathcal{G}_s(f_k) \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ for every $k \in \mathbb{N}$, we have $g \equiv 0$ on $\partial\Omega$. Thus, introducing the function $v_0 : \mathbb{R}^n \to \mathbb{R}$ defined by

$$v_0(x) := \begin{cases} g(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

from (2.5) (and since $\mathcal{G}_s(f_k) \equiv v_0 \equiv 0$ on $\mathbb{R}^n \setminus \Omega$), we get

$$\lim_{k\to\infty} \|\mathcal{G}_s(f_k) - v_0\|_{L^\infty(\mathbb{R}^n)} = \lim_{k\to\infty} \|\mathcal{G}_s(f_k) - v_0\|_{L^\infty(\Omega)} = 0,$$

which is precisely the desired (2.4). This ends the proof. □

Then, we prove that $\mathcal{G}_s$ is *positive* with respect to the cone

$$\mathcal{C} := \big\{ f \in L^\infty(\Omega) : \ f \geq 0 \text{ a.e. on } \Omega \big\}. \tag{2.6}$$

Proposition 2.5. *If $\mathcal{C}$ is as in (2.6), then $\mathcal{G}_s\big(\mathcal{C}\big) \subseteq \mathcal{C}$.*

Proof. Let $f \in \mathcal{C}$ be fixed, and let $u_f := \mathcal{G}_s(f)$. By definition, u_f is the unique solution of (2.1) in $H^s(\mathbb{R}^n) \cap C^{0,s}(\mathbb{R}^n)$; thus, since $f \geq 0$ a.e. on Ω (as $f \in \mathcal{C}$), from the Weak Maximum Principle and the continuity of u_f, we get

$$u_f = \mathcal{G}_s(f) \geq 0 \text{ pointwise in } \mathbb{R}^n.$$

This ends the proof. □

We close this section by briefly studying the *spectrum* of $\mathcal{G}_s$. To this end, we first recall a result on the eigenvalues of $(-\Delta)^s$, which follows by combining [27, Proposition 9] and [24, Corollary 1.6] (see also [26, Proposition 4]).

Theorem 2.6. *There exists a countable set $\Lambda \subseteq (0,\infty)$ such that, if $\lambda \in \Lambda$, there exists a solution $e_\lambda \in H^s(\mathbb{R}^n) \cap C^{0,s}(\mathbb{R}^n)$ of the eigenvalue problem*

$$\begin{cases} (-\Delta)^s u = \lambda\, u & \text{in } \Omega, \\ u \equiv 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

This means, precisely, that $e_\lambda \equiv 0$ in $\mathbb{R}^n \setminus \Omega$ and

$$\frac{c_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{\big(e_\lambda(x) - e_\lambda(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x-y|^{n+2s}}\, \mathrm{d}x\, \mathrm{d}y = \lambda \int_\Omega e_\lambda\, \varphi\, \mathrm{d}x$$

for every test function $\varphi \in C_0^\infty(\Omega)$.

Thanks to Theorem 2.6, we can prove the following proposition.

Proposition 2.7. *Let $r(\mathcal{G}_s)$ denote the spectral radius of $\mathcal{G}_s$, thought of as an operator from $L^\infty(\Omega)$ into itself. Then, the following facts hold:*

(i) $r(\mathcal{G}_s) > 0$;
(ii) there exists a non-negative function $\phi \in C^{0,s}(\mathbb{R}^n) \setminus \{0\}$ such that

$$\mathcal{G}_s\big(\phi|_\Omega\big) = r(\mathcal{G}_s)\,\phi \qquad \text{and} \qquad \phi \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega.$$

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Proof.

(i) On account of Theorem 2.6, we can find a real $\lambda > 0$ and a function $e_\lambda \in C^{0,s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, not identically vanishing, such that

$$\begin{cases} (-\Delta)^s e_\lambda = \lambda\, e_\lambda & \text{in } \Omega, \\ e_\lambda \equiv 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

As a consequence, since $e_\lambda|_\Omega \in L^\infty(\Omega)$ (as $e_\lambda \in C^{0,s}(\mathbb{R}^n)$), we have

$$\mathcal{G}_s(e_\lambda|_\Omega) = \frac{1}{\lambda}\, e_\lambda,$$

and this proves that $r(\mathcal{G}_s) > 0$, as desired.

(ii) We prove the assertion by using the well-known Krein–Rutman theorem. To this end, we first observe that, if C is the cone defined in (2.6), one has:
   (a) $C - C$ is dense in $L^\infty(\Omega)$ (actually, $C - C = L^\infty(\Omega)$);
   (b) $\mathcal{G}_s(C) \subseteq C$ (see Corollary 2.5).

Moreover, from (i) we know that $r(\mathcal{G}_s) > 0$. Thus, since Proposition 2.4 ensures that $\mathcal{G}_s$ is compact from $L^\infty(\Omega)$ into itself, we can invoke Krein–Rutman's theorem, ensuring that $\lambda = r(\mathcal{G}_s)$ is an eigenvalue of $\mathcal{G}_s$ with a corresponding eigenfunction in C. This means that there exists a nontrivial function $\phi \in C$, such that

$$\mathcal{G}_s(\phi) = r(\mathcal{G}_s)\, \phi. \tag{2.7}$$

On the other hand, since $\mathcal{G}_s(\phi) = u_\phi$ belongs to $C^{0,s}(\mathbb{R}^n)$ and vanishes on $\mathbb{R}^n \setminus \Omega$ (reminding that u_ϕ is the solution of (2.1)), from (2.7), we conclude that

$$\phi = \frac{1}{r(\mathcal{G}_s)}\, u_\phi \in C^{0,s}(\mathbb{R}^n) \qquad \text{and} \qquad \phi \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega.$$

This ends the proof. □

2.2 | The nonhomogeneous Dirichlet problem for $(-\Delta)^s$

Due to its relevance in the sequel, we spend a few words about the nonhomogeneous Dirichlet problem for $(-\Delta)^s$, that is,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u \equiv \zeta & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.8}$$

Avoiding to discuss the solvability of (2.8) for general f and ζ, here we limit to observe that, when ζ is sufficiently regular, problem (2.8) can be reduced to (2.1) (with a different f). In fact, let us suppose that

$$\zeta \in H^s(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \cap C^2(\overline{\mathcal{O}}),$$

where $\mathcal{O} \subseteq \mathbb{R}^n$ is some open neighborhood of $\overline{\Omega}$. Then, it is not difficult to see that $(-\Delta)^s\zeta$ can be computed pointwise in Ω, and

$$(-\Delta)^s\zeta(x) = -\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{\zeta(x+z) + \zeta(x-z) - 2\zeta(x)}{|z|^{n+2s}}\, \mathrm{d}z \quad (x \in \Omega).$$

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In particular, $(-\Delta)^s\zeta \in L^\infty(\Omega)$. Moreover, since $\zeta \in H^s(\mathbb{R}^n)$, a standard "integration-by-parts" argument gives, for every test function $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega (-\Delta)^s\zeta \cdot \varphi \, dx = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big(\zeta(x)-\zeta(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}} \, dx\, dy.$$

On account of these facts, we obtain the following result.

Theorem 2.8. *Let $f \in L^\infty(\Omega)$ and let $\zeta \in H^s(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \cap C^2(\overline{\mathcal{O}})$, where $\mathcal{O} \subseteq \mathbb{R}^n$ is an open neighborhood of $\overline{\Omega}$. Then, the function*

$$u_{f,\zeta} := \mathcal{G}_s\big(f - (-\Delta)^s\zeta\big) + \zeta \in H^s(\mathbb{R}^n) \cap C(\mathbb{R}^n)$$

is the unique (weak) solution of (2.8). This means, precisely, that

$$\frac{c_{n,s}}{2} \iint_{\mathbb{R}^{2n}} \frac{(u_{f,\zeta}(x)-u_{f,\zeta}(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} \, dx\, dy = \int_\Omega f\varphi \, dx \quad \forall\ \varphi \in C_0^\infty(\Omega),$$

and $u_{f,\zeta} \equiv \zeta$ pointwise on $\mathbb{R}^n \setminus \Omega$.

Remark 2.9. We explicitly notice that, since f and $(-\Delta)^s\zeta$ are in $L^\infty(\Omega)$, by the very definition of $\mathcal{G}_s$ one has

$$v := \mathcal{G}_s\big(f - (-\Delta)^s\zeta\big) \in C^{0,s}(\mathbb{R}^n);$$

as a consequence, if $\zeta \in C^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1)$, we derive that

$$u_{f,\zeta} = v + \zeta \in C^{0,\theta}(\mathbb{R}^n), \qquad \text{where } \theta := \min\{s,\alpha\}.$$

2.3 | The fixed point index

For the sake of completeness, we collect in the following proposition some properties of the classical fixed point index that will be crucial in the proof of our existence result; for more details, see, for example, [4, 17].

In what follows, the closure and the boundary of subsets of a cone $\hat{P}$ are understood to be relative to $\hat{P}$.

Proposition 2.10. *Let X be a real Banach space and let $\hat{P} \subset X$ be a cone. Let D be an open bounded set of X with $0 \in D \cap \hat{P}$ and $\overline{D \cap \hat{P}} \neq \hat{P}$. Assume that $T : \overline{D \cap \hat{P}} \to \hat{P}$ is a compact operator such that $x \neq T(x)$ for $x \in \partial(D \cap \hat{P})$. Then, the fixed point index $i_{\hat{P}}(T, D \cap \hat{P})$ has the following properties:*

(i) If there exists $e \in \hat{P} \setminus \{0\}$ such that $x \neq T(x) + \sigma e$ for all $x \in \partial(D \cap \hat{P})$ and all $\sigma > 0$, then $i_{\hat{P}}(T, D \cap \hat{P}) = 0$.
(ii) If $T(x) \neq \sigma x$ for all $x \in \partial(D \cap \hat{P})$ and all $\sigma > 1$, then $i_{\hat{P}}(T, D \cap \hat{P}) = 1$.
(iii) Let D^1 be an open bounded subset of X such that $\overline{(D^1 \cap \hat{P})} \subset (D \cap \hat{P})$. If $i_{\hat{P}}(T, D \cap \hat{P}) = 1$ and $i_{\hat{P}}(T, D^1 \cap \hat{P}) = 0$, then T has a fixed point in $(D \cap \hat{P}) \setminus \overline{(D^1 \cap \hat{P})}$.
The same holds if $i_{\hat{P}}(T, D \cap \hat{P}) = 0$ and $i_{\hat{P}}(T, D^1 \cap \hat{P}) = 1$.

3 | EXISTENCE OF POSITIVE SOLUTIONS

In this section, we state and prove our main existence result for positive solutions of (1.1), namely, Theorem 3.3 below. Before doing this, we fix the relevant "structural" assumptions, which shall be tacitly understood in the sequel.

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As already mentioned in the Introduction, our aim is to prove the existence of positive solutions for Dirichlet problems of the following form:

$$\begin{cases} (-\Delta)^{s_i} u_i = \lambda_i f_i(x, \mathbf{u}, \mathcal{P}_i[\mathbf{u}]) & \text{in } \Omega \qquad (i = 1, \dots, m), \\ u_i(x) = \eta_i \zeta_i(x) \mathcal{B}_i[\mathbf{u}], & \text{for } x \in \mathbb{R}^n \setminus \Omega \quad (i = 1, \dots, m). \end{cases} \tag{3.1}$$

Here, $m \geq 1$ is a fixed integer, $\mathbf{u} = (u_1, \dots, u_m)$, with $u_1, \dots, u_m : \mathbb{R}^n \to \mathbb{R}$, $s_1, \dots, s_m \in (0,1)$ and, for every fixed $i \in \{1, \dots, m\}$, we assume that

(H0) $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with smooth boundary;
(H1) f_i is a real-valued function defined on $\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}$;
(H2) $\mathcal{P}_i$, $\mathcal{B}_i$ are real-valued operators acting on the space

$$\mathbb{X} := \left\{ u \in C(\mathbb{R}^n; \mathbb{R}^m) : \sup_{\mathbb{R}^n} |u_i| < \infty \ \text{ for all } i = 1, \dots, m. \right\}; \tag{3.2}$$

(H3) $\zeta_i \in H^s(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \cap C^2(\overline{\mathcal{O}})$ and $\zeta_i \geq 0$ on $\mathbb{R}^n$, where

$$s = \min_{i=1,\dots,m} s_i, \tag{3.3}$$

and $\mathcal{O} \subseteq \mathbb{R}^n$ is a suitable open neighborhood of $\overline{\Omega}$;
(H4) λ_i, η_i are nonnegative parameters.

In dealing with vector-valued functions $\mathbf{u} = (u_1, \dots, u_m) : \mathbb{R}^n \to \mathbb{R}^m$, it is more convenient to use on the spaces $\mathbb{R}^p$ (for $p \in \mathbb{N}$) the *maximum norm*, that is,

$$\|z\| := \max_{j=1,\dots,p} |z_j| \qquad \text{for all } z \in \mathbb{R}^p.$$

Thus, if $\mathbb{X}$ is as in (3.2) and $\mathbf{u} \in \mathbb{X}$, we define

$$\|\mathbf{u}\|_\infty := \sup_{x \in \mathbb{R}^n} \|\mathbf{u}(x)\| = \max_{i=1,\dots,m} \left(\sup_{\mathbb{R}^n} |u_i(x)| \right).$$

Obviously, $(\mathbb{X}, \|\cdot\|_\infty)$ is a (real) Banach space.

Now, we have already anticipated that we aim to study the solvability of problem (3.1) by means of suitable fixed-point techniques; on the other hand, since the "non-local" BCs are prescribed on the complementary of Ω, one should work in the space $C(\mathbb{R}^n; \mathbb{R}^m)$, which *is not* a Banach space.

To overcome this issue, we make the following key observation: if $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^m$ is a (continuous) function solving (3.1) pointwise in $\mathbb{R}^n$, then

$$u_i(x) = \eta_i \zeta_i(x) \mathcal{B}_i[\mathbf{u}] \quad \text{for all } x \text{ on } \mathbb{R}^n \setminus \Omega \qquad (i = 1, \dots, m);$$

as a consequence, since $\mathbf{u}$ is continuous on $\mathbb{R}^n$ and $\zeta_i \in C_b(\mathbb{R}^n)$, we deduce that $\mathbf{u} \in \mathbb{X}$, and $(\mathbb{X}, \|\cdot\|_\infty)$ *is* a Banach space. We then use the results in Section 2 to define a suitable functional $\mathcal{T}$, acting on the space $\mathbb{X}$, allowing us to rephrase problem (3.1) into the fixed-point equation $\mathcal{T}(\mathbf{u}) = \mathbf{u}$ in $\mathbb{X}$.

To begin with, if $i \in \{1, \dots, m\}$ is fixed, for the sake of simplicity, we denote by $\mathcal{G}^i$ the $(-\Delta)^{s_i}$-Green operator $\mathcal{G}_{s_i}$ defined in (2.3). Let us recall that

$$\mathcal{G}^i : L^\infty(\Omega) \to C^{0,s_i}(\mathbb{R}^n)$$

and, by definition, $\mathcal{G}^i(\mathfrak{f}) = u_\mathfrak{f}$ is the unique solution in $C^{0,s_i}(\mathbb{R}^n)$ of

$$\begin{cases} (-\Delta)^{s_i} u = \mathfrak{f} & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

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(with $\mathfrak{f} \in L^\infty(\Omega)$). In particular, $\mathcal{G}^i(\mathfrak{f}) \equiv 0$ on $\mathbb{R}^n \setminus \Omega$. As a consequence, if $s \in (0,1)$ is as in (3.3), we derive that

$$\mathcal{G}^i\big(L^\infty(\Omega)\big) \subseteq L^\infty(\mathbb{R}^n) \cap C^{0,s}(\mathbb{R}^n) \qquad \text{for all } i = 1, \dots, m. \tag{3.4}$$

According to Proposition 2.7, we then let $r_i = r(\mathcal{G}^i) > 0$ be the spectral radius of $\mathcal{G}^i$, thought of as an operator from $L^\infty(\Omega)$ into itself, and we fix once and for all a nonnegative function $\phi_i \in C^{0,s}(\mathbb{R}^n) \setminus \{0\}$ such that (setting $\mu_i := 1/r_i$)

$$\phi_i = \mu_i \, \mathcal{G}^i\big(\phi_i|_\Omega\big) \qquad \text{and} \qquad \phi_i \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega. \tag{3.5}$$

To proceed further, we define the *Nemytskii* operator $\mathcal{F}_i$ as

$$\mathcal{F}_i(\mathbf{u}) = f_i(\cdot, \mathbf{u}(\cdot), \mathcal{P}_i[\mathbf{u}]) \qquad (\mathbf{u} \in \mathbb{X}),$$

and we assume for a moment that, for any $\mathbf{u} \in \mathbb{X}$, we have $\mathcal{F}_i(\mathbf{u}) \in L^\infty(\Omega)$; we will prove later that this holds under the assumptions of Theorem 3.3. In this case, taking into account (3.4), we can set

$$\mathcal{I}(\mathbf{u}) := \Big(\lambda_i \, \mathcal{G}^i\big(\mathcal{F}_i(\mathbf{u})\big)\Big)_{i=1,\dots,m} \in \mathbb{X}.$$

Furthermore, if $\gamma_i \in H^s(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is the unique solution of

$$\begin{cases} (-\Delta)^{s_i} u = 0 & \text{in } \Omega, \\ u \equiv \zeta_i & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{3.6}$$

(according to Theorem 2.8), given $\mathbf{u} \in \mathbb{X}$ we define

$$\mathcal{D}(\mathbf{u}) := \Big(\eta_i \, \gamma_i(\cdot) \, \mathcal{B}_i[\mathbf{u}]\Big)_{i=1,\dots,m}.$$

We explicitly notice that, since $\gamma_i \in C(\mathbb{R}^n)$, $\zeta_i \in C_b(\mathbb{R}^n)$, and $\gamma_i \equiv \zeta_i$ on $\mathbb{R}^n \setminus \Omega$, we have $\gamma_i \in \mathbb{X}$; thus, since $\mathcal{D}(\mathbf{u})$ is a scalar multiple of γ_i, one has $\mathcal{D}(\mathbf{u}) \in \mathbb{X}$.

Using the operators $\mathcal{I}$ and $\mathcal{D}$ just introduced, we can finally provide the precise definition of *solution* of problem (3.1).

Definition 3.1. We say that a function $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{X}$ is a *solution* of problem (3.1) if it satisfies the following properties:

(i) $\mathcal{F}_i(\mathbf{u}) \in L^\infty(\Omega)$ for every $i = 1, \dots, m$;
(ii) $\mathbf{u} = \mathcal{I}(\mathbf{u}) + \mathcal{D}(\mathbf{u})$, that is,

$$u_i = \lambda_i \, \mathcal{G}^i\big(\mathcal{F}_i(\mathbf{u})\big) + \eta_i \, \gamma_i(\cdot) \, \mathcal{B}_i[\mathbf{u}].$$

If, in addition, $u_i \geq 0$ for all $i = 1, \dots, m$ and there exists some $i_0 \in \{1, \dots, m\}$ such that $u_{i_0} \not\equiv 0$, we say that $\mathbf{u}$ is a *nonzero positive solution* of (3.1).

Remark 3.2. On account of Remark 2.9, if $\zeta_1, \dots, \zeta_m \in C^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1]$, we have that $\gamma_i = u_{0,\zeta_i} \in C^{0,\theta}(\mathbb{R}^n)$ for all $i = 1, \dots, m$, where $\theta = \min\{\alpha, s\}$ and s is as in (3.3). As a consequence, we have $\mathcal{D}(\mathbb{X}) \subseteq C^{0,\theta}(\mathbb{R}^n; \mathbb{R}^m)$ and, by (3.4), any solution of problem (3.1) actually belongs to $C^{0,\theta}(\mathbb{R}^n; \mathbb{R}^m)$.

For our existence result, we make use of the classical fixed point index (see Section 2.3). We will work on the cone

$$P := \Big\{\mathbf{u} \in \mathbb{X} : \ u_i \geq 0 \text{ on } \mathbb{R}^n \text{ for every } i = 1, \dots, m\Big\}.$$

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Given a finite sequence $\varrho = \{\rho_i\}_{i=1}^m \subseteq (0, +\infty)$, we define

$$I(\varrho) = \prod_{i=1}^{m} [0, \rho_k] \subseteq \mathbb{R}^m \tag{3.7}$$

and set

$$P(\varrho) := \Big\{ \mathbf{u} \in \mathbb{X} : \ \mathbf{u}(x) \in I(\varrho) \text{ for all } x \in \mathbb{R}^n \Big\}. \tag{3.8}$$

Theorem 3.3. *Let the assumptions (H0) to (H4) be in force. Moreover, let us suppose that there exists $\varrho = \{\rho_i\}_{i=1}^m \subseteq (0, \infty)$ for which the following hypotheses are satisfied:*

(a) For every $i = 1, \dots, m$, one has that
(a)$_1$ *$\mathcal{P}_i\big|_{P(\varrho)}$ is continuous, and there exist $\underline{\omega}_{i,\varrho}, \overline{\omega}_{i,\varrho} \in \mathbb{R}$ such that*

$$\underline{\omega}_{i,\varrho} \le \mathcal{P}_i[\mathbf{u}] \le \overline{\omega}_{i,\varrho}, \quad \text{for every } \mathbf{u} \in P(\varrho);$$

(a)$_2$ *f_i is continuous and nonnegative on $\overline{\Omega} \times I(\varrho) \times [\underline{\omega}_{i,\varrho}, \overline{\omega}_{i,\varrho}]$;*
(a)$_3$ *$\mathcal{B}_i\big|_{P(\varrho)}$ is continuous, nonnegative, and bounded.*

(b) There exist $\delta \in (0, +\infty)$, $i_0 \in \{1, 2, \dots, m\}$, and $\rho_0 \in (0, \min_i \rho_i)$ such that, setting $\varrho_0 := \{\rho_0\}_{i=1}^m$, we have

$$f_{i_0}(x, \mathbf{z}, \omega) \ge \delta z_{i_0} \quad \text{for every } (x, \mathbf{z}, \omega) \in \Pi_0, \tag{3.9}$$

where $\Pi_0 := \overline{\Omega} \times I(\varrho_0) \times [\underline{\omega}_0, \overline{\omega}_0]$ and

$$\underline{\omega}_0 := \inf_{\mathbf{u} \in P(\varrho_0)} \mathcal{P}_{i_0}[\mathbf{u}], \qquad \overline{\omega}_0 := \sup_{\mathbf{u} \in P(\varrho_0)} \mathcal{P}_{i_0}[\mathbf{u}].$$

(c) Setting, for every $i = 1, \dots, m$,

$$\begin{aligned} M_i &:= \max \Big\{ f_i(x, \mathbf{z}, \omega) : (x, \mathbf{z}, \omega) \in \overline{\Omega} \times I(\varrho) \times [\underline{\omega}_{i,\varrho}, \overline{\omega}_{i,\varrho}] \Big\} \quad \text{and} \\ B_i &:= \sup_{\mathbf{u} \in P(\varrho)} \mathcal{B}_i[\mathbf{u}], \end{aligned} \tag{3.10}$$

the following inequalities are satisfied:
(c)$_1$ *$\mu_{i_0} \le \delta \lambda_{i_0}$;*
(c)$_2$ *$\lambda_i M_i \, \|\mathcal{G}^i(\hat{1})\|_\infty + \eta_i B_i \|\gamma_i\|_\infty \le \rho_i$.*

Then, the system (3.1) has a nonzero positive solution $\mathbf{u} \in \mathbb{X}$ such that

$$\|\mathbf{u}\|_\infty \ge \rho_0 \qquad \text{and} \qquad \|u_i\|_\infty \le \rho_i \text{ for every } i = 1, \dots, m. \tag{3.11}$$

Proof. As a preliminary fact, we explicitly observe that $\mathcal{T} := \mathcal{I} + \mathcal{D}$ is *a well-defined operator* from $P(\varrho)$ to $\mathbb{X}$. For this purpose, we show that

$$\mathcal{F}_i\big(P(\varrho)\big) \subseteq L^\infty(\Omega) \qquad (\text{for every } i = 1, \dots, m). \tag{3.12}$$

In fact, given $\mathbf{u} \in P(\varrho)$, by assumption (a)$_1$ we have, for every $i = 1, \dots, m$,

$$(x, \mathbf{u}(x), \mathcal{P}_i[\mathbf{u}]) \in \overline{\Omega} \times I(\varrho) \times [\underline{\omega}_{i,\varrho}, \overline{\omega}_{i,\varrho}] \quad \text{for all } x \in \overline{\Omega};$$

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as a consequence, from assumption $(a)_2$ we readily derive (3.12). Notice that $\mathcal{T}$ maps $P(\varrho)$ into $\mathbb{X}$ in view of (3.4) and the very definition of $\mathcal{D}$.

We now show that the following assertions hold:

(i) $\mathcal{T}$ maps $P(\varrho)$ into $P \subseteq \mathbb{X}$;
(ii) $\mathcal{T} : P(\varrho) \to P$ is compact.

To prove (i), let $\mathbf{u} \in P(\varrho)$ and let $i \in \{1, \dots, m\}$ be fixed. We first observe that, since $\mathbf{u} \in P(\varrho)$, by $(a)_3$ we have $\mathcal{B}_i[\mathbf{u}] \geq 0$; moreover, since $\zeta_i \geq 0$ on $\mathbb{R}^n$ (see assumption (H3)), from the Weak Maximum Principle we derive that $\gamma_i \geq 0$. Thus, the fact that η_i is nonnegative implies that

$$\mathcal{D}(\mathbf{u})_i = \eta_i \, \gamma_i(\cdot) \, \mathcal{B}_i[\mathbf{u}] \geq 0 \quad \text{on } \mathbb{R}^n.$$

On the other hand, since $\mathbf{u} \in P(\varrho)$, by assumption $(a)_2$ we also have

$$\mathcal{F}_i(\mathbf{u}) = f_i(\cdot, \mathbf{u}(\cdot), \mathcal{P}_i[\mathbf{u}]) \geq 0;$$

as a consequence, from Corollary 2.5, we infer that $\mathcal{G}^i\big(\mathcal{F}_i(\mathbf{u})\big) \geq 0$ on $\mathbb{R}^n$, and thus, as $\lambda_i \geq 0$, we get

$$\mathcal{I}(\mathbf{u})_i = \lambda_i \mathcal{G}^i(\mathcal{F}_i(\mathbf{u})) \geq 0 \quad \text{on } \mathbb{R}^n.$$

Gathering together these facts, and bearing in mind the very definition of $\mathcal{T}$, we conclude that $\mathcal{T}(P(\varrho)) \subseteq P$, as claimed.

We now prove assertion (ii). To this end, we fix $i \in \{1, \dots, m\}$ and we observe that, in view of assumption $(a)_2$, we have that $\mathcal{F}_i(\mathbf{u}) : P(\varrho) \to L^\infty(\Omega)$ is continuous; moreover, by Proposition 2.4 one also has that

$$\mathcal{G}^i : L^\infty(\Omega) \to L^\infty(\mathbb{R}^n)$$

is linear and compact. As a consequence, we deduce that

$$\mathcal{I} = (\lambda_i \, \mathcal{G}^i \circ \mathcal{F}_i)_{i=1,\dots,m} : P(\varrho) \to L^\infty(\mathbb{R}^n; \mathbb{R}^m)$$

is compact. On the other hand, since $\mathcal{D}$ is bounded on $P(\varrho)$ (as the same is true of $\mathcal{B}_1, \dots, \mathcal{B}_m$, see assumption $(a)_3$) and since, by definition,

$$\mathcal{D}(P(\varrho)) \subseteq \big\{ t\boldsymbol{\gamma} : t \in \mathbb{R} \big\} \subseteq L^\infty(\mathbb{R}^n; \mathbb{R}^m) \qquad (\text{where } \boldsymbol{\gamma} := (\gamma_1, \dots, \gamma_m)),$$

we get that $\mathcal{T} = \mathcal{I} + \mathcal{D}$ is compact from $P(\varrho)$ into $L^\infty(\mathbb{R}^n; \mathbb{R}^m)$. From this, since $\mathcal{T}(P(\varrho)) \subseteq P$ (by assertion (i)) and since P is a closed subspace of $L^\infty(\mathbb{R}^n; \mathbb{R}^m)$, we conclude that $\mathcal{T}$ is compact from $P(\varrho)$ into P.

To proceed further, we define

$$P_0 = \Big\{ \mathbf{u} \in \mathbb{X} : \ \mathbf{u}(x) \in I(\varrho_0) \text{ for all } x \in \mathbb{R}^n \Big\} \subseteq P(\varrho) \subseteq P,$$

where ϱ_0 is as in assumption (b). Moreover, we consider the open sets

$$D := \big\{ \mathbf{u} \in \mathbb{X} : \ \|u_i\|_\infty < \rho_i \text{ for every } i = 1, \dots, m \big\} \qquad \text{and}$$

$$D^1 := \big\{ \mathbf{u} \in \mathbb{X} : \ \|u_i\|_\infty < \rho_0 \text{ for every } i = 1, \dots, m \big\}.$$

We explicitly observe that, if D, D^1 are as above, we have $\partial(D \cap P) = \partial P(\varrho)$ and $\partial(D^1 \cap P) = \partial P_0$, where both the boundaries are relative to P.

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Now, if the operator $\mathcal{T}$ has a fixed point $\mathbf{u}_0 \in \partial P(\varrho) \cup \partial P_0$, then $\mathbf{u}_0$ is a solution of problem (3.1) satisfying (3.11), and the theorem is proved. If, instead, $\mathcal{T}$ is fixed-point free on $\partial P(\varrho) \cup \partial P_0$, both the fixed-point indices

$$i_P(\mathcal{T}, D \cap P) \qquad \text{and} \qquad i_P(\mathcal{T}, D^1 \cap P)$$

are well defined. Assuming this last possibility, we prove the following.

CLAIM 1. We claim that

$$i_P(\mathcal{T}, D \cap P) = 1. \tag{3.13}$$

According to Proposition 2.10-(ii), to prove (3.13), it suffices to show that

$$\mathcal{A}(\mathbf{u}) \neq \sigma\, \mathbf{u} \qquad \text{for every } \mathbf{u} \in \partial P(\varrho) \text{ and every } \sigma > 1, \tag{3.14}$$

To establish (3.14), we argue by contradiction, and we suppose that there exist $\mathbf{u} \in \partial P(\varrho)$ and $\sigma > 1$ such that

$$\sigma \mathbf{u} = \mathcal{T}(\mathbf{u}).$$

Since $\mathbf{u} \in \partial P(\varrho)$, there exists an index $i \in \{1, \dots, m\}$ such that $\|u_i\|_\infty = \rho_i$. By assumption $(a)_1$ and (3.10), we have

$$0 \leq \mathcal{F}_i(\mathbf{u})(x) = f_i(x, \mathbf{u}(x), \mathcal{P}_i[\mathbf{u}]) \leq M_i \qquad \text{for all } x \in \overline{\Omega};$$

so that

$$\begin{aligned} \sigma u_i(x) &= \lambda_i\, \mathcal{G}^i\big(\mathcal{F}_i(\mathbf{u})\big)(x) + \eta_i\, \gamma_i(x)\, \mathcal{B}_i[\mathbf{u}] \\ &\leq \lambda_i\, \mathcal{G}^i\big(M_i \hat{1}\big)(x) + \eta_i\, \gamma_i(x)\, \mathcal{B}_i[\mathbf{u}] \\ &\leq \big\|\lambda_i\, \mathcal{G}^i\big(M_i \hat{1}\big)\big\|_\infty + \big\|\eta_i\, B_i\, \gamma_i\big\|_\infty \\ &= \lambda_i\, M_i\, \big\|\mathcal{G}^i(\hat{1})\big\|_\infty + \eta_i\, B_i\, \|\gamma_i\|_\infty \leq \rho_i \end{aligned} \tag{3.15}$$

the last inequality following by assumption $(c)_2$. As a consequence, taking the supremum for $x \in \overline{\Omega}$ in (3.15) and recalling that $\mathbf{u} \in \partial P(\varrho) \subseteq P(\varrho)$, we get

$$\sup_{x \in \overline{\Omega}} |\sigma\, u_i(x)| \leq \sigma\, \rho_i \leq \rho_i,$$

which is clearly a contradiction (since $\sigma > 1$), and the claim is proved.

CLAIM 2. We claim that

$$i_P(\mathcal{T}, D^1 \cap P) = 0. \tag{3.16}$$

According to Proposition 2.10-(i), to prove (3.16), it suffices to show that there exists a suitable function $e \in P \setminus \{0\}$ satisfying the property

$$\mathcal{T}(\mathbf{u}) + \sigma e \neq \mathbf{u} \qquad \text{for every } \mathbf{u} \in \partial P_0 \text{ and every } \sigma > 0. \tag{3.17}$$

To establish (3.17), we let $e := (\phi_1, \dots, \phi_m)$, where each component $\phi_1, \dots, \phi_m$ is as in (3.5), and we argue by contradiction: We thus suppose that there exist $\mathbf{u} \in \partial P_0$ and $\sigma > 0$ such that

$$\mathbf{u} = \mathcal{T}(\mathbf{u}) + \sigma e.$$

Let i_0 be as in assumption (b). Since $\mathcal{T}(\mathbf{u}) \in P$, we have

$$u_{i_0} = \mathcal{T}(\mathbf{u})_{i_0} + \sigma\, \phi_{i_0} \geq \sigma \phi_{i_0} \qquad \text{on } \overline{\Omega}.$$

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Furthermore, again by assumption (b), for every $x \in \overline{\Omega}$, we get

$$\mathcal{F}_{i_0}(\mathbf{u})(x) = f_{i_0}(x, \mathbf{u}(x), \mathcal{P}_{i_0}[\mathbf{u}]) \geq \delta u_{i_0}(x) \geq \delta\sigma\phi_{i_0}(x).$$

Gathering together all these facts, for every $x \in \overline{\Omega}$, we have

$$\begin{aligned} u_{i_0}(x) &= \lambda_{i_0}\mathcal{G}^{i_0}\big(\mathcal{F}_{i_0}(\mathbf{u})\big)(x) + \eta_{i_0}\,\gamma_{i_0}(x)\,\mathcal{B}_{i_0}[\mathbf{u}] + \sigma\,\phi_{i_0}(x) \\ &\geq \lambda_{i_0}\,\mathcal{G}^{i_0}(\delta\sigma\phi_{i_0})(x) + \sigma\phi_{i_0}(x) \\ &= \frac{\delta\lambda_{i_0}}{\mu_{i_0}} \cdot \sigma\phi_{i_0}(x) + \sigma\phi_{i_0}(x) \geq 2\sigma\phi_{i_0}(x) \end{aligned}$$

the last inequality following by assumption $(c)_1$. By iterating the above argument, for every $x \in \overline{\Omega}$ we get

$$u_{i_0}(x) \geq p\sigma\phi_{i_0}(x) \qquad \text{for every } p \in \mathbb{N},$$

which is in contradiction with the fact that u_{i_0} is bounded.

We are now ready to conclude the proof of the theorem: In fact, by combining Claims 1 and 2 and Proposition 2.10-(iii), we infer the existence of a fixed point

$$\mathbf{u}_0 \in \big(D \cap P\big) \setminus P_0$$

of $\mathcal{T}$; thus, $\mathbf{u}_0$ is a solution of (3.1) satisfying (3.11). □

An elementary argument yields the following nonexistence result.

Theorem 3.4. *Let the assumptions (H0) to (H4) be in force. Moreover, let us suppose that there exists $\varrho = \{\rho_i\}_{i=1}^m \subseteq (0, \infty)$ such that, for every $i = 1, \dots, m$, the following conditions hold:*

(a) there exist $\underline{\omega}_{i,\varrho}$, $\overline{\omega}_{i,\varrho} \in \mathbb{R}$ such that

$$\underline{\omega}_{i,\varrho} \leq \mathcal{P}_i[\mathbf{u}] \leq \overline{\omega}_{i,\varrho} \quad \text{for every } \mathbf{u} \in P(\varrho);$$

(b) there exist $\tau_i \in (0, +\infty)$ such that

$$f_i(x, \mathbf{z}, \omega) \leq \tau_i z_i \quad \text{for every } (x, \mathbf{z}, \omega) \in \overline{\Omega} \times I(\varrho) \times [\underline{\omega}_{i,\varrho}, \overline{\omega}_{i,\varrho}];$$

(c) there exist $\xi_i \in (0, +\infty)$ such that

$$\big|\mathcal{B}_i[\mathbf{u}]\big| \leq \xi_i \cdot \|\mathbf{u}\|_\infty, \quad \text{for every } u \in P(\varrho);$$

(d) the following inequality holds:

$$\lambda_i\tau_i\,\|\mathcal{G}^i(\hat{1})\|_\infty + \eta_i\,\xi_i\|\gamma_i\|_\infty < 1. \tag{3.18}$$

Then, the system (3.1) has at most the zero solution in $P(\varrho)$.

Proof. By contradiction, assume that (3.1) has a solution $\mathbf{u} \in P(\varrho) \setminus \{0\}$, that is, for every $i = 1, \dots, m$ we have

$$\mathcal{F}_i(\mathbf{u}) \in L^\infty(\Omega) \quad \text{and} \quad u_i = \lambda_i\,\mathcal{G}^i\big(\mathcal{F}_i(\mathbf{u})\big) + \eta_i\,\gamma_i(\cdot)\mathcal{B}_i[\mathbf{u}].$$

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Setting $\rho := \|\mathbf{u}\|_\infty > 0$, we let $j \in \{1, 2, \dots, m\}$ be such that

$$\|u_j\|_\infty = \rho. \tag{3.19}$$

In view of assumptions (a) and (b), for every $x \in \overline{\Omega}$, we then have

$$\mathcal{F}_j(\mathbf{u})(x) = f_j(x, \mathbf{u}(x), \mathcal{P}_j[\mathbf{u}]) \le \tau_j u_j(x) \le \tau_j \rho,$$

and thus (see Corollary 2.5, and recalling that $\mathcal{F}_j(\mathbf{u}) \in L^\infty(\Omega)$)

$$\mathcal{G}^j\big(\tau_j \rho \cdot \hat{1} - \mathcal{F}_j(\mathbf{u})\big) \ge 0 \iff \mathcal{G}^j(\mathcal{F}_j(\mathbf{u})) \le \tau_j \rho\, \mathcal{G}^j(\hat{1}) \qquad \text{on } \mathbb{R}^n.$$

As a consequence, we obtain

$$\begin{aligned} u_j(x) &= \lambda_j\, \mathcal{G}^j\big(\mathcal{F}_j(\mathbf{u})\big)(x) + \eta_j\, \gamma_j(x)\, \mathcal{B}_j[\mathbf{u}] \\ &\le \lambda_j \tau_j \rho\, \mathcal{G}^j(\hat{1})(x) + \eta_j\, \gamma_j(x)\, \mathcal{B}_j[\mathbf{u}] \\ &\quad \text{(by assumption (c) and (3.19))} \\ &\le \|\lambda_j \tau_j \rho\, \mathcal{G}^j(\hat{1})\|_\infty + \big\|\eta_j\, \xi_j \rho\, \gamma_j\big\|_\infty \\ &= \big(\lambda_j\, \tau_j\, \|\mathcal{G}^j(\hat{1})\|_\infty + \eta_j\, \xi_j\, \|\gamma_j\|_\infty\big)\rho. \end{aligned} \tag{3.20}$$

By taking the supremum in (3.20) for $x \in \overline{\Omega}$, from (3.18) and (3.19) we get

$$\rho = \sup_{x \in \overline{\Omega}} u_j(x) \le \big(\lambda_j\, \tau_j\, \|\mathcal{G}^j(\hat{1})\|_\infty + \eta_j\, \xi_j\, \|\gamma_j\|_\infty\big)\rho < \rho,$$

and this is clearly a contradiction. Thus, we conclude that problem (3.1) cannot have nonzero solutions in $P(\varrho)$, and the proof is complete. □

4 | EXAMPLES

In this last section, we present a couple of concrete examples illustrating the applicability of Theorems 3.3 and 3.4. Before proceeding, we remind the following result, which shall play a key role in our computations.

Lemma 4.1. *Let $r > 0$ be fixed, and let $B_r \subseteq \mathbb{R}^n$ be the Euclidean ball centered at 0 with radius r. Moreover, let $s \in (0, 1)$. Then, the unique solution v_s of*

$$\begin{cases} (-\Delta)^s v = 1 & \text{in } B_r, \\ v \equiv 0 & \text{on } \mathbb{R}^n \setminus B_r \end{cases}$$

has the following explicit expression

$$v_s(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma(\frac{n+2s}{2})\,\Gamma(1+s)}\,\big(r^2 - \|x\|^2\big)_+^s,$$

where $\|\cdot\|$ stands for the usual Euclidean norm and

$$\Gamma(\alpha) = \int_{\mathbb{R}} x^{\alpha-1}\, e^{-x}\, \mathrm{d}x \qquad (\alpha > 0).$$

For a proof of Lemma 4.1 we refer, for example, to [10, 16].

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Example 4.2. In Euclidean space $\mathbb{R}^2$, let us consider the following BVP:

$$\begin{cases} (-\Delta)^{\frac{1}{4}} u_1 = \lambda_1 (1-u_1) \int_{B_1} e^{u_2}\,\mathrm{d}x & \text{in } B_1, \\ (-\Delta)^{\frac{3}{4}} u_2 = \lambda_2 u_2 \cdot \mathrm{osc}_{B_1}(u_1) & \text{in } B_1, \\ u_1\big|_{\mathbb{R}^2\setminus B_1} = \eta_1 \cdot u_1(0)u_2(0), \\ u_2\big|_{\mathbb{R}^2\setminus B_1} = \eta_2 \cdot \limsup\limits_{|x|\to\infty} u_1(x), \end{cases} \tag{4.1}$$

where B_1 is the Euclidean ball centered at 0 with radius 1, and

$$\mathrm{osc}_{B_1}(\phi) := \sup_{B_1}(\phi) - \inf_{B_1}(\phi) \qquad (\text{for all } \phi \in \mathbb{X}). \tag{4.2}$$

Clearly, problem (4.1) is of the form (3.1), with

1. $\Omega = B_1$, $m = 2$, $s_1 = 1/4$ and $s_2 = 3/4$;
2. $f_1 : \overline{B}_1 \times \mathbb{R}^2 \times \mathbb{R}, \quad f_1(x, \mathbf{z}, w) := (1 - z_1)\, w$;
3. $f_2 : \overline{B}_1 \times \mathbb{R}^2 \times \mathbb{R}, \quad f_2(x, \mathbf{z}, w) := z_2\, w$;
4. $\mathcal{P}_1 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{P}_1[\mathbf{u}] := \int_{B_1} e^{u_2}\,\mathrm{d}x$;
5. $\mathcal{P}_2 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{P}_2[\mathbf{u}] := \mathrm{osc}_{B_1}(u_1)$;
6. $\mathcal{B}_1 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{B}_1[\mathbf{u}] := u_1(0)u_2(0)$;
7. $\mathcal{B}_2 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{B}_2[\mathbf{u}] := \limsup_{|x|\to\infty} u_1(x)$;
8. $\zeta_1 \equiv \zeta_2 \equiv 1$.

Moreover, it is straightforward to recognize that all the "structural" assumptions (H0) to (H4) listed at the beginning of Section 3 are fulfilled. We now turn to prove that, in this case, also assumptions (a) to (c) of Theorem 3.3 are satisfied for a suitable choice of the nonnegative parameters $\lambda_1, \lambda_2, \eta_1, \eta_2$.

To this end, we consider the (finite) sequence ϱ defined as follows:

$$\varrho = \{\rho_1, \rho_2\}, \qquad \text{where } \rho_1 := \frac{1}{2} \text{ and } \rho_2 := 1. \tag{4.3}$$

According to this choice of ϱ, we have (see (3.7) and (3.8))

$$I(\varrho) = [0, 1/2] \times [0, 1], \qquad P(\varrho) = \left\{ u \in \mathbb{X} : \ \mathbf{u}(x) \in I(\varrho) \text{ for all } x \in \mathbb{R}^2 \right\}.$$

Assumption (a). First, notice that $\mathcal{P}_1, \mathcal{P}_2$ are continuous when restricted to $P(\varrho) \subseteq \mathbb{X}$; moreover, for every $\mathbf{u} = (u_1, u_2) \in P(\varrho)$, we have

$$\pi \le \mathcal{P}_1[\mathbf{u}] = \int_{B_1} e^{u_2}\,\mathrm{d}x \le \pi \cdot e \quad \text{and} \quad 0 \le \mathcal{P}_2[\mathbf{u}] = \mathrm{osc}(u_1) \le \frac{1}{2},$$

so that assumption $(\mathrm{a})_1$ is satisfied with the choices

$$\underline{\omega}_{1,\varrho} := \pi, \quad \overline{\omega}_{1,\varrho} := \pi \cdot e, \quad \underline{\omega}_{2,\varrho} := 0, \quad \overline{\omega}_{2,\varrho} := 1/2. \tag{4.4}$$

As regards assumption $(\mathrm{a})_2$, we first notice that $f_1, f_2 \in C(\overline{B}_1 \times \mathbb{R}^2 \times \mathbb{R})$; moreover, by taking into account (4.3) and (4.4), we get

$$f_1(x, \mathbf{z}, w) = (1 - z_1)\, w \ge \frac{\pi}{2} > 0 \ \text{ on } \overline{B_1} \times I(\varrho) \times [\pi, \pi \cdot e] \qquad \text{and}$$

$$f_2(x, \mathbf{z}, w) = z_2\, w \ge 0 \ \text{ on } \overline{B_1} \times I(\varrho) \times [0, 1/2],$$

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so that $(a)_2$ is fulfilled. Finally, as regards assumption $(a)_3$, it is not difficult to check that $\mathcal{B}_1, \mathcal{B}_2$ are continuous and nonnegative when restricted to $P(\varrho)$; moreover, since for every $\mathbf{u} = (u_1, u_2) \in P(\varrho)$, we have

$$|\mathcal{B}_1[\mathbf{u}]| = |u_1(0)u_2(0)| \le \frac{1}{2} \quad \text{and} \quad |\mathcal{B}_2[\mathbf{u}]| = |\limsup_{|x|\to\infty} u_1(x)| \le \frac{1}{2},$$

we conclude that $\mathcal{B}_1, \mathcal{B}_2$ are bounded on $P(\varrho)$.

Assumption (b). First of all, if $\rho_0 \in (0, 1/2)$ is arbitrarily fixed, we have

$$\underline{\omega}_0 := \inf_{\mathbf{u}\in P(\varrho_0)} \mathcal{P}_1[\mathbf{u}] = \pi,$$

where $\varrho_0 := \{\rho_0, \rho_0\}$; moreover, for every $(x, \mathbf{z}) \in \overline{B}_1 \times I(\varrho_0)$ and every $w \ge \pi$, one has

$$f_1(x, \mathbf{z}, w) = (1 - z_1)\, w \ge \frac{\pi}{2}.$$

Gathering together these facts, we conclude that (3.9) holds *for every choice of* $\delta > 0$. In fact, given any such δ, we define

$$\rho_0 = \rho_0(\delta) := \min\left\{\frac{1}{4}, \frac{\pi}{2\delta}\right\} \in (0, 1/2); \tag{4.5}$$

then, for every $(x, \mathbf{z}) \in \overline{B}_1 \times I(\varrho_0)$ and every $w \ge \pi = \underline{\omega}_0$, we get

$$f_1(x, \mathbf{z}, w) \ge \frac{\pi}{2\delta} \cdot \delta \ge \delta z_1,$$

and thus assumption (b) is satisfied with $i_0 = 1$ (and for every $\delta > 0$).

Assumption (c). We start by computing the constants appearing in (3.10). First of all, using (4.3), (4.4), and the definition of f_1, f_2, we get

$$\begin{aligned} M_1 &= \sup\left\{f_1(x, \mathbf{z}, w) : (x, \mathbf{z}, w) \in \overline{B}_1 \times I(\varrho) \times [\underline{\omega}_{1,\varrho}, \overline{\omega}_{1,\varrho}]\right\} = \pi \cdot e \quad \text{and} \\ M_2 &= \sup\left\{f_2(x, \mathbf{z}, w) : (x, \mathbf{z}, w) \in \overline{B}_1 \times I(\varrho) \times [\underline{\omega}_{2,\varrho}, \overline{\omega}_{2,\varrho}]\right\} = \frac{1}{2}. \end{aligned} \tag{4.6}$$

Moreover, again by (4.3), (4.4), and the definition of $\mathcal{B}_1, \mathcal{B}_2$, we get

$$B_1 = \sup_{\mathbf{u}\in P(\varrho)} \mathcal{B}_1[\mathbf{u}] = \frac{1}{2} \quad \text{and} \quad B_2 = \sup_{\mathbf{u}\in P(\varrho)} \mathcal{B}_2[\mathbf{u}] = \frac{1}{2}. \tag{4.7}$$

We then turn our attention to the functions $\mathcal{G}^i(\hat{1}) = \mathcal{G}_{s_i}(\hat{1})$ and γ_i (for $i = 1, 2$). To begin with, by definition of $(-\Delta)^{s_i}$-Green operator, we know that $\mathcal{G}^i(\hat{1})$ is the unique solution in $C^{0,s_i}(\mathbb{R}^2)$ of

$$\begin{cases} (-\Delta)^{s_i} v = 1 & \text{in } B_1, \\ v \equiv 0 & \text{on } \mathbb{R}^2 \setminus B_1. \end{cases}$$

On the other hand, thanks to Lemma 4.1, we can write *the explicit expression* of $\mathcal{G}^i(\hat{1})$: More precisely, since in our context we have $n = 2, r = 1, s_1 = 1/4$, and $s_2 = 3/4$, we get

$$\mathcal{G}^1(\hat{1}) = \mathcal{G}_{1/4}(\hat{1}) = \left(\frac{2^{-1/4}}{\Gamma(5/4)}\right)^2 (1 - \|x\|^2)_+^{1/4} \qquad \text{and}$$

$$\mathcal{G}^2(\hat{1}) = \mathcal{G}_{3/4}(\hat{1}) = \left(\frac{2^{-3/4}}{\Gamma(7/4)}\right)^2 (1 - \|x\|^2)_+^{3/4}.$$

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As a consequence, we obtain

$$\begin{aligned} \|\mathcal{G}^1(\hat{1})\|_\infty &= \frac{1}{\sqrt{2}\,\Gamma^2(5/4)} \approx 0.860682 \qquad \text{and} \\ \|\mathcal{G}^2(\hat{1})\|_\infty &= \frac{1}{\sqrt{8}\,\Gamma^2(7/4)} \approx 0.418567. \end{aligned} \tag{4.8}$$

As for the functions γ_i, the computations are much more easier: First of all, since $\zeta_1 \equiv \zeta_2 \equiv 1$, we know from (3.6) that γ_i is the unique solution of

$$\begin{cases} (-\Delta)^{s_i} v = 0 & \text{in } B_1, \\ u \equiv 1 & \text{on } \mathbb{R}^2 \setminus B_1. \end{cases}$$

On the other hand, since the above problem is solved by the constant function $\hat{\gamma} \equiv 1$ (independently of s_i), we get $\gamma_1 \equiv \gamma_2 \equiv 1$. Hence, we have

$$\|\gamma_1\|_\infty = \|\gamma_2\|_\infty = 1. \tag{4.9}$$

Gathering together *all the facts established so far*, we are finally in a position to apply Theorem 3.3: taking into account (4.3), (4.6), (4.7), (4.8), and (4.9), for every choice of parameters $\lambda_1, \lambda_2, \eta_1, \eta_2 \geq 0$ satisfying

$$\begin{aligned} &\lambda_1 \frac{\pi \cdot e}{\sqrt{2}\,\Gamma^2(5/4)} + \frac{\eta_1}{2} \leq \frac{1}{2} && \text{(see assumption (c)}_2 \text{ with } i = 1), \\ &\frac{\lambda_2}{2} \frac{1}{\sqrt{8}\,\Gamma^2(7/4)} + \frac{\eta_2}{2} \leq 1 && \text{(see assumption (c)}_2 \text{ with } i = 2), \\ &\lambda_1 > 0 && \text{(see assumption (c)}_1 \text{ with } i_0 = 1), \end{aligned} \tag{4.10}$$

there exists a solution $\mathbf{u}_0 \in C^{0,1/4}(\mathbb{R}^2)$ of (4.1), further satisfying

$$\|\mathbf{u}\|_\infty \geq \rho_0(\delta) \qquad \text{and} \qquad \|u_1\|_\infty \leq \frac{1}{2}, \ \|u_2\|_\infty \leq 1.$$

Here, $\rho(\delta) \in (0, 1/2)$ is as in (4.5) and $\delta > 0$ is chosen in such a way that

$$\delta\lambda_1 \geq \mu_1, \tag{4.11}$$

where μ_1 is the inverse of the spectral radius of $(-\Delta)^{1/4}$. More explicitly, given $\lambda_1, \lambda_2, \eta_1, \eta_2$ satisfying (4.10), one *first chooses* $\delta > 0$ in such a way that $\delta\lambda_1 \geq \mu_1$ (see assumption $(c)_1$); *then*, one lets $\rho_0 = \rho_0(\delta)$ be as in (4.5).

The key point in this argument is that, since (3.9) holds *for every* $\delta > 0$ (by accordingly choosing ρ_0), one is free to choose $\delta > 0$ in such a way that (4.11) holds (provided that $\lambda_1 > 0$), without the need of an explicit knowledge of μ_1.

Example 4.3. In Euclidean space $\mathbb{R}^2$, we consider the following BVP:

$$\begin{cases} (-\Delta)^{\frac{1}{4}} u_1 = \lambda_1 u_1^2 (1 - u_1) \int_{B_1} e^{u_2}\, dx & \text{in } B_1, \\ (-\Delta)^{\frac{3}{4}} u_2 = \lambda_2 u_2 \cdot \operatorname{osc}_{B_1}(u_1) & \text{in } B_1, \\ u_1|_{\mathbb{R}^2 \setminus B_1} = \eta_1 \cdot u_1(0) u_2(0), \\ u_2|_{\mathbb{R}^2 \setminus B_1} = \eta_2 \cdot \limsup_{|x| \to \infty} u_1(x), \end{cases} \tag{4.12}$$

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where B_1 is the Euclidean unit ball, and $\operatorname{osc}_{B_1}(\cdot)$ is as in (4.2). Clearly, problem (4.12) is of the form (3.1), with

1. $\Omega = B_1$, $m = 2$, $s_1 = 1/4$, and $s_2 = 3/4$;
2. $f_1 : \overline{B}_1 \times \mathbb{R}^2 \times \mathbb{R}, \quad f_1(x, \mathbf{z}, w) := z_1^2(1 - z_1)\, w$;
3. $f_2 : \overline{B}_1 \times \mathbb{R}^2 \times \mathbb{R}, \quad f_2(x, \mathbf{z}, w) := z_2\, w$;
4. $\mathcal{P}_1 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{P}_1[\mathbf{u}] := \int_{B_1} e^{u_2}\, \mathrm{d}x$;
5. $\mathcal{P}_2 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{P}_2[\mathbf{u}] := \operatorname{osc}_{B_1}(u_1)$;
6. $\mathcal{B}_1 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{B}_1[\mathbf{u}] := u_1(0)u_2(0)$;
7. $\mathcal{B}_2 : \mathbb{X} \to \mathbb{R}, \quad \mathcal{B}_2[\mathbf{u}] := \limsup_{|x|\to\infty} u_1(x)$;
8. $\zeta_1 \equiv \zeta_2 \equiv 1$.

Moreover, it is straightforward to recognize that all the "structural" assumptions (H0) to (H4) listed at the beginning of Section 3 are fulfilled. We now aim to show that, despite the similarity between problems (4.12) and (4.1), in this case, it is possible to choose the parameters $\lambda_1, \lambda_2, \eta_1, \eta_2$ in such a way that assumptions (a) to (d) of the *nonexistence* Theorem 3.4 are satisfied.

To this end, we consider the (finite) sequence ϱ defined as

$$\varrho := \{\rho_1, \rho_2\}, \qquad \text{where } \rho_1 = \rho_2 = 1; \tag{4.13}$$

According to this choice of ϱ, we have (see (3.7) and (3.8))

$$I(\varrho) = [0,1] \times [0,1], \qquad P(\varrho) = \left\{u \in \mathbb{X} : \ \mathbf{u}(x) \in I(\varrho) \text{ for all } x \in \mathbb{R}^2\right\}.$$

Assumption (a). Given any $\mathbf{u} = (u_1, u_2) \in P(\varrho)$, we have

$$\pi \le \mathcal{P}_1[\mathbf{u}] = \int_{B_1} e^{u_2}\, \mathrm{d}x \le \pi \cdot e \quad \text{and} \quad 0 \le \mathcal{P}_2[\mathbf{u}] = \operatorname{osc}(u_1) \le 1;$$

hence, assumption (a) is fulfilled with the choices

$$\underline{\omega}_{1,\varrho} := \pi, \quad \overline{\omega}_{1,\varrho} := \pi \cdot e, \quad \underline{\omega}_{2,\varrho} := 0, \quad \overline{\omega}_{2,\varrho} := 1. \tag{4.14}$$

Assumption (b). We first observe that both f_1 and f_2 are continuous on the whole of $\overline{B}_1 \times \mathbb{R}^2 \times \mathbb{R}$; moreover,

$$0 \le f_1(x, \mathbf{z}, w) = z_1^2(1 - z_1)w \le (\pi \cdot e)z_1 \quad \text{on } \overline{B}_1 \times I(\varrho) \times [\pi, \pi \cdot e]$$
$$\text{and} \quad 0 \le f_2(x, \mathbf{z}, w) = z_2 w \le z_2 \quad \text{on } \overline{B}_1 \times I(\varrho) \times [0, 1].$$

Thus, assumption (b) is satisfied with the choices

$$\tau_1 = \pi \cdot e \qquad \text{and} \qquad \tau_2 = 1. \tag{4.15}$$

Assumption (c). Given any $\mathbf{u} = (u_1, u_2) \in P(\varrho)$, we see that

$$0 \le \mathcal{B}_1[\mathbf{u}] = u_1(0)u_2(0) \le u_1(0) \le \|\mathbf{u}\|_\infty \qquad \text{and}$$
$$0 \le \mathcal{B}_2[\mathbf{u}] = \limsup_{|x|\to\infty} u_1(x) \le \|\mathbf{u}\|_\infty.$$

Hence, assumption (c) is satisfied with the choices

$$\xi_1 = \xi_2 = 1. \tag{4.16}$$

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Assumption (d). First of all, by exploiting all the computations carried out in Example 4.2 (see, respectively, (4.8) and (4.9)), we know that

(a) $\|\mathcal{G}^1(\hat{1})\|_\infty = \frac{1}{\sqrt{2}\,\Gamma^2(5/4)}$ and $\|\mathcal{G}^2(\hat{1})\|_\infty = \frac{1}{\sqrt{8}\,\Gamma^2(7/4)}$;
(b) $\|\gamma_1\|_\infty = \|\gamma_2\|_\infty = 1$.

As a consequence, by gathering together (4.15), (4.16), and the above (a) and (b), we can apply Theorem 3.4: for every choice of $\lambda_1, \lambda_2, \eta_1, \eta_2 \geq 0$ satisfying

$$\lambda_1 \frac{\pi \cdot e}{\sqrt{2}\,\Gamma^2(5/4)} + \eta_1 < 1 \qquad \text{and} \qquad \frac{\lambda_2}{\sqrt{8}\,\Gamma^2(7/4)} + \eta_2 < 1,$$

the BVP (4.12) possesses only the zero solution in $P(\varrho)$ (notice that the constant function $\mathbf{u} \equiv 0$ is indeed a solution of (4.12)).

ACKNOWLEDGMENTS

The authors were partially supported by G.N.A.M.P.A. - INdAM (Italy). G. Infante is a member of the UMI Group TAA "Approximation Theory and Applications."

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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REFERENCES

[1] C. O. Alves, R. N. de Lima, and A. B. Nóbrega, *Bifurcation properties for a class of fractional Laplacian equations in* $\mathbb{R}^N$, Math. Nachr. **291** (2018), no. 14-15, 2125–2144. https://doi.org/10.1002/mana.201700284.
[2] C. O. Alves, R. N. de Lima, and A. B. Nóbrega, *Global bifurcation results for a fractional equation in* $\mathbb{R}^N$, J. Math. Anal. Appl. **487** (2020), no. 1, 123980, 21. https://doi.org/10.1016/j.jmaa.2020.123980.
[3] C. O. Alves and O. H. Miyagaki, *Existence and concentration of solution for a class of fractional elliptic equation in* $\mathbb{R}^N$ *via penalization method*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, Art. 47, 19. https://doi.org/10.1007/s00526-016-0983-x.
[4] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), no. 4, 620–709. https://doi.org/10.1137/1018114.
[5] H. Amann and M. G. Crandall, *On some existence theorems for semi-linear elliptic equations*, Indiana Univ. Math. J. **27** (1978), no. 5, 779–790. https://doi.org/10.1512/iumj.1978.27.27050.
[6] V. Ambrosio, *Concentrating solutions for a class of nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, Rev. Mat. Iberoam. **35** (2019), no. 5, 1367–1414. https://doi.org/10.4171/rmi/1086.
[7] V. Ambrosio and T. Isernia, *A multiplicity result for a fractional Kirchhoff equation in* $\mathbb{R}^N$ *with a general nonlinearity*, Commun. Contemp. Math. **20** (2018), no. 5, 1750054, 17. https://doi.org/10.1142/S0219199717500547.
[8] V. Ambrosio and R. Servadei, *Supercritical fractional Kirchhoff type problems*, Fract. Calc. Appl. Anal. **22** (2019), no. 5, 1351–1377. https://doi.org/10.1515/fca-2019-0071.
[9] S. Biagi, A. Calamai, and G. Infante, *Nonzero positive solutions of elliptic systems with gradient dependence and functional BCs*, Adv. Nonlinear Stud. **20** (2020), no. 4, 911–931. https://doi.org/10.1515/ans-2020-2101.
[10] K. Bogdan, T. Grzywny, and M. Ryznar, *Heat kernel estimates for the fractional Laplacian with Dirichlet conditions*, Ann. Probab. **38** (2010), no. 5, 1901–1923. https://doi.org/10.1214/10-AOP532.
[11] W. Chen, *Multiplicity of solutions for a fractional Kirchhoff type problem*, Commun. Pure Appl. Anal. **14** (2015), no. 5, 2009–2020. https://doi.org/10.3934/cpaa.2015.14.2009.
[12] A. C. R. Costa and B. B. V. Maia, *Multiple solutions of systems involving fractional Kirchhoff-type equations with critical growth*, Differ. Equ. Appl. **12** (2020), no. 2, 165–184. https://doi.org/10.7153/dea-2020-12-11.
[13] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004.

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[14] G. Estrada-Rodriguez, H. Gimperlein, K. J. Painter, and J. Stocek, *Space-time fractional diffusion in cell movement models with delay*, Math. Models Methods Appl. Sci. **29** (2019), no. 1, 65–88. https://doi.org/10.1142/S0218202519500039.

[15] A. Fiscella and E. Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal. **94** (2014), 156–170. https://doi.org/10.1016/j.na.2013.08.011.

[16] R. K. Getoor, *First passage times for symmetric stable processes in space*, Trans. Amer. Math. Soc. **101** (1961), 75–90. https://doi.org/10.2307/1993412.

[17] D. J. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, Notes and Reports in Mathematics in Science and Engineering, vol. 5, pp. viii+275, Academic Press, Inc., Boston, MA, 1988.

[18] T. Hillen and A. Buttenschön, *Nonlocal adhesion models for microorganisms on bounded domains*, SIAM J. Appl. Math. **80** (2020), no. 1, 382–401. https://doi.org/10.1137/19M1250315.

[19] G. Infante, *Nonzero positive solutions of a multi-parameter elliptic system with functional BCS*, Topol. Methods Nonlinear Anal. **52** (2018), no. 2, 665–675. https://doi.org/10.12775/tmna.2017.060.

[20] G. Infante, *Nonzero positive solutions of nonlocal elliptic systems with functional BCs*, J. Elliptic Parabol. Equ. **5** (2019), no. 2, 493–505. https://doi.org/10.1007/s41808-019-00049-6.

[21] G. Infante, *Eigenvalues of elliptic functional differential systems via a Birkhoff–Kellogg type theorem*, Mathematics **9** (2020), no. 1, 4. https://doi.org/10.3390/math9010004.

[22] G. Lemon and J. R. King, *A functional differential equation model for biological cell sorting due to differential adhesion*, Math. Models Methods Appl. Sci. **23** (2013), no. 1, 93–126. https://doi.org/10.1142/S0218202512500467.

[23] B. Perthame, W. Sun, and M. Tang, *The fractional diffusion limit of a kinetic model with biochemical pathway*, Z. Angew. Math. Phys. **69** (2018), no. 3, Paper No. 67, 15. https://doi.org/10.1007/s00033-018-0964-3.

[24] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. (9) **101** (2014), no. 3, 275–302. https://doi.org/10.1016/j.matpur.2013.06.003.

[25] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, J. Math. Phys. **54** (2013), no. 3, 031501, 17. https://doi.org/10.1063/1.4793990.

[26] R. Servadei and E. Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal. **12** (2013), no. 6, 2445–2464. https://doi.org/10.3934/cpaa.2013.12.2445.

[27] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), no. 5, 2105–2137. https://doi.org/10.3934/dcds.2013.33.2105.

[28] G. Sun and K. Teng, *Existence and multiplicity of solutions for a class of fractional Kirchhoff-type problem*, Math. Commun. **19** (2014), no. 1, 183–194.

How to cite this article: S. Biagi, A. Calamai, and G. Infante, *Nonzero positive solutions of fractional Laplacian systems with functional terms*, Math. Nachr. **296** (2023), 102–121. https://doi.org/10.1002/mana.202100074

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