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ORIGINAL ARTICLE

A precise asymptotic description of half-linear differential equations

Pavel Řehák

Institute of Mathematics, FME, Brno University of Technology, Brno, Czech Republic

Correspondence
Pavel Řehák, Institute of Mathematics, FME, Brno University of Technology, Czech Republic.
Email: rehak.pavel@fme.vutbr.cz

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Abstract

We study asymptotic behavior of solutions of nonoscillatory second-order half-linear differential equations. We give (in some sense optimal) conditions that guarantee generalized regular variation of all solutions, where no sign condition on the potential is assumed. For all of these solutions, we establish precise asymptotic formulas, where positive as well as negative potential is considered. We examine, as consequences, also equations with regularly varying coefficients, or with the coefficients viewed as perturbations of exponentials, or the equations under certain critical (double roots) settings. We make also asymptotic analysis of Poincaré–Perron solutions. Many of our results are new even in the linear case.

KEYWORDS
asymptotic formula, half-linear differential equation, nonoscillatory solution, Poincaré–Perron solution, regular variation

1 | INTRODUCTION

Consider the second-order half-linear differential equation of the form

$$(r(t)\Phi(u'))' + p(t)\Phi(u) = 0 \tag{1}$$

for $t \in [a, \infty)$, $a > 0$, where $r > 0$ and p are continuous functions on $[a, \infty)$, $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$, $\alpha > 1$. By Φ^{-1}, we mean the inverse of Φ. Note that $\Phi^{-1}(u) = |u|^{\beta-1} \operatorname{sgn} u$, where β is the conjugate number to α, that is, $1/\alpha + 1/\beta = 1$. Our principal purpose is to provide precise information on asymptotic behavior of all elements of its solutions space (which is homogeneous but not additive) in nonoscillatory case. We rely mainly on the theory of regular variation and on various transformations.

It is well known that the theory of regular variation (see Section 2) provides powerful tools for asymptotic analysis of differential equations, see, in particular, the monograph [19] by Marić published in 2000 and the text [28], which maps the situation after Marić's book. As for half-linear differential equations, they have been studied in the framework of regular variation quite extensively, see [7, 12–16, 22, 23, 25, 28–33]. While some of the papers focused on showing the existence of regularly varying solutions under as general conditions as possible, other papers tried to establish more or less precise asymptotic formulas and/or dealt with the entire special classes of solutions (if not with the entire solution space) under

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additional conditions. Various delicate settings (such as the critical case for generalized Euler equations) were treated too. In addition to the classical Karamata theory of regular variation, also the de Haan theory (in particular, rapid variation and the classes Gamma and Pi) found applications there. Of course, asymptotics of half-linear equations were studied also in other frameworks than regular variation, see, for example, [2–4, 6–8, 10, 17, 18, 20, 21, 24, 34, 35]; as it can be seen, some of these works are pertinent also to our theory.

The purpose of our new paper is to make the analysis, which, along with the results in [32], fills existing gaps in previous results and provides a somehow complete picture on asymptotics of half-linear equations. Besides showing regular variation of all solutions and establishing precise formulas for them, we discuss also relations with other classes such as those constructed according to the limit values of solutions and their quasiderivatives or the classes of principal solutions (the last relation is treated in [32]). We also offer a revision of the previous results and discuss critical settings. Further, we utilize our results to analyze Poincaré–Perron solutions, which are associated to perturbations of autonomous half-linear equations. Our observations are important also from the stability point of view and in the study of radially symmetric solutions to partial differential equations with p-Laplacian.

We emphasize that in the statements where regular variation of all solutions is shown, no sign condition on p is needed. Moreover, in the statements where p is required to be eventually of one sign, both cases, positive p and negative p, are considered. Many of our results and observations are new even in the linear case. This concerns practically all the results where $p > 0$ or p is not of one sign and also the results where $p < 0$ under a critical setting (the term critical is clarified below) or with p, r general (note that some of these results appeared already earlier in author's work but are included also here for completeness). We should emphasize that, for example, while the results such as Theorem 6 can be obtained in the linear case in a substantially simpler way, namely, via the linear substitution $y = hu$ in Equation (1) with $\alpha = 2$, the situation in the general half-linear case is much more complicated since this "linear tool" is not at disposal due to the lack of additivity of the solution space.

The asymptotic formulas are derived for all (generalized) regularly varying solutions. However, it should be stressed that we work under the conditions, which guarantee (generalized) regular variation of all solutions to Equation (1) (where regular variation of eventually negative solutions is meant as taking their absolute values). Thus, we provide the description of the entire solution space of (1). In addition, under our setting, the sufficient conditions for regular variation become also necessary, so they are optimal.

The paper is organized as follows. In the next section, we recall the concept of regularly varying functions including their properties that are needed in our analysis. Basic information on nonoscillatory solutions of (1) are given in Section 3. In Section 4, we present the results under general setting: First, we deal with (generalized) regular variation of the entire solution space of (1) and then we establish asymptotic formulas distinguishing several logical cases that are exhaustively examined. Asymptotics of equations with regularly varying coefficients is studied in Section 5, while equations with perturbed exponential coefficients are treated in Section 6. The critical (or double root) setting is discussed in Section 7. In Section 8, we derive asymptotic formulas for the so-called Poincaré–Perron solutions. All proofs are gathered in Section 9.

2 | REGULAR VARIATION

A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying (at infinity) of index* ϑ (see [1, 9]) if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for every } \lambda \in (0, \infty); \tag{2}$$

we write $f \in \mathcal{RV}(\vartheta)$. If $\vartheta = 0$, we speak about *slowly varying* functions; we write $f \in \mathcal{SV}$, thus $\mathcal{SV} = \mathcal{RV}(0)$. If $f \in \mathcal{RV}(\vartheta)$, then relation (2) holds uniformly on each compact λ-set in $(0, \infty)$ (this is the so-called *Uniform Convergence Theorem*, see, for example, [1]). It follows that $f \in \mathcal{RV}(\vartheta)$ if and only if there exists a function $L \in \mathcal{SV}$ such that $f(t) = t^{\vartheta} L(t)$ for every t. The slowly varying component of $f \in \mathcal{RV}(\vartheta)$ will be denoted by L_f, that is,

$$L_f(t) := \frac{f(t)}{t^{\vartheta}}, \tag{3}$$

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unless stated otherwise. We adopt notation (3) also for negative functions f such that $|f| \in \mathcal{RV}(\vartheta)$. The so-called *Representation Theorem* (see, e.g., [1]) says the following: $f \in \mathcal{RV}(\vartheta)$ if and only if

$$f(t) = \varphi(t)t^{\vartheta} \exp\left\{\int_a^t \frac{\psi(s)}{s}\,\mathrm{d}s\right\}, \tag{4}$$

$t \geq a$, for some $a > 0$, where φ, ψ are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0,\infty)$ and $\lim_{t\to\infty} \psi(t) = 0$. A function $f \in \mathcal{RV}(\vartheta)$ can alternatively be represented as

$$f(t) = \varphi(t) \exp\left\{\int_a^t \frac{\xi(s)}{s}\,\mathrm{d}s\right\}, \tag{5}$$

$t \geq a$, for some $a > 0$, where φ, ξ are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0,\infty)$ and $\lim_{t\to\infty} \xi(t) = \vartheta$. A regularly varying function f is said to be *normalized regularly varying*, we write $f \in \mathcal{NRV}(\vartheta)$, if $\varphi(t) \equiv C$ in (4) or in (5). If (4) holds with $\vartheta = 0$ and $\varphi(t) \equiv C$, we say that f is *normalized slowly varying*, we write $f \in \mathcal{NSV}$. The basic theory of regularly varying functions is sometimes called the Karamata theory. Selected properties of $\mathcal{RV}$ functions are gathered in Proposition 1 and Theorem 7

The following type of extension of $\mathcal{RV}$ functions was introduced in [11]. The motivation was primarily for purposes of studying differential equations. Consider a continuously differentiable function H, which is positive and satisfies $H'(t) > 0$ for $t \in [b,\infty)$ and $\lim_{t\to\infty} H(t) = \infty$. A measurable function $f : [a,\infty) \to (0,\infty)$ is called *regularly varying of index ϑ with respect to H* if $f \circ H^{-1} \in \mathcal{RV}(\vartheta)$; we write $f \in \mathcal{RV}_H(\vartheta)$. If $\vartheta = 0$, then f is called *slowly varying with respect to H*; we write $f \in \mathcal{SV}_H$. As an immediate consequence of the properties of $\mathcal{RV}$ functions, we have: $f \in \mathcal{RV}_H(\vartheta)$ if and only if $f(t) = H^{\vartheta}(t)L^{[H]}(t)$, where $L^{[H]} \in \mathcal{SV}_H$. The slowly varying (with respect to H) component of $f \in \mathcal{RV}_H(\vartheta)$ will be denoted by $L_f^{[H]}$, that is,

$$L_f^{[H]}(t) := \frac{f(t)}{H^{\vartheta}(t)};$$

we adopt this notation also for negative functions f such that $|f| \in \mathcal{RV}_H(\vartheta)$. The *Generalized Representation Theorem* says that $L^{[H]} \in \mathcal{SV}_H$ if and only if

$$L^{[H]}(t) = \varphi(t) \exp\left\{\int_a^t \frac{H'(s)}{H(s)}\psi(s)\,\mathrm{d}s\right\},$$

$t \geq a$, for some $a > 0$, where φ, ψ are measurable and $\varphi(t) \to C \in (0,\infty)$, $\psi(t) \to 0$ as $t \to \infty$. If $\varphi(t) \equiv C$, then $L^{[H]}$ is called *normalized slowly varying with respect to H*; we write $L^{[H]} \in \mathcal{NSV}_H$. A generalized regularly varying function $f(t) = H^{\vartheta}(t)L^{[H]}(t)$ with $L^{[H]} \in \mathcal{NSV}_H$ is called *normalized regularly varying* of index ϑ with respect to H; we write $f \in \mathcal{NRV}_H(\vartheta)$. Selected properties of generalized $\mathcal{RV}$ functions are gathered in Proposition 3.

A measurable function $f : [a,\infty) \to \mathbb{R}$ is said to belong to the class Π (see [1, 9]) if there exists a function $w : (0,\infty) \to (0,\infty)$ such that for $\lambda > 0$,

$$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{w(t)} = \ln\lambda; \tag{6}$$

we write $f \in \Pi$ or $f \in \Pi(w)$. The function w is called an *auxiliary function* for f. The class Π, after taking absolute values, forms a proper subclass of $\mathcal{SV}$. The class Π, along with some other classes, belong among the so-called de Haan classes. Selected properties of Π functions are gathered in Proposition 2. One can extend the notion of Π class similarly as it was done for $\mathcal{RV}$ class in the preceding paragraph. Indeed, let H be the same as above. Then, we write $f \in \Pi_H(w)$ if and only if $f \circ H^{-1} \in \Pi(w)$.

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The natural and useful generalization of regular variation is the concept of regularly bounded functions (or O-regularly varying functions). A measurable function $f : [a,\infty) \to (0,\infty)$ is called *regularly bounded* if

$$0 < \liminf_{t\to\infty} \frac{f(\lambda t)}{f(t)} \le \limsup_{t\to\infty} \frac{f(\lambda t)}{f(t)} < \infty$$

for every $\lambda > 1$; we write $f \in \mathcal{RB}$. The representation theorem for $\mathcal{RB}$ functions reads as follows: $f \in \mathcal{RB}$ if and only if

$$f(t) = \exp\left\{\xi(t) + \int_a^t \frac{\eta(s)}{s}\,\mathrm{d}s\right\}, \tag{7}$$

$t \ge a$, where ξ and η are bounded and measurable on $[a,\infty)$. If $\xi(t) \equiv C \in \mathbb{R}$, then f is referred to as *normalized regularly bounded function*; we write $f \in \mathcal{NRB}$. The concept of rapid variation deals with the natural problem when the limit in (2) attains its extreme values. A measurable function $f : [a,\infty) \to (0,\infty)$ is called *rapidly varying of index* ∞ (resp. $-\infty$) if $\lim_{t\to\infty} f(\lambda t)/f(t) = 0$ (resp. ∞) for $0 < \lambda < 1$ and $\lim_{t\to\infty} f(\lambda t)/f(t) = \infty$ (resp. 0) for $\lambda > 1$; we write $f \in \mathcal{RPV}(\infty)$ (resp. $f \in \mathcal{RPV}(-\infty)$). If f is positive and differentiable and there exists $\lim_{t\to\infty} tf'(t)/f(t) = \pm\infty$, then $f \in \mathcal{RPV}(\pm\infty)$.

3 | NONOSCILLATORY SOLUTIONS

This paper can be understood as a certain complement to [32], accordingly we recall basic facts on nonoscillatory solutions and introduce the same relevant notation as in [32].

Since we are primarily interested in regularly varying solutions of (1), we work with nonoscillatory solutions of (1), that is, with the solutions that are eventually positive or eventually negative. Note that if y is a solution of (1), then $-y$ is also a solution of (1) because of the homogeneity of the solution space. Recall that by the Sturm-type separation theorem, which extends to half-linear equations, see [7, chapter 1], a solution of (1) is oscillatory (i.e., it is not of eventually one sign) if and only if all solutions of (1) are oscillatory. Hence, we can classify Equation (1) as oscillatory or nonoscillatory as in the linear case. In fact, we are interested in behavior of all nonoscillatory solutions of (1). Since the solution space (1) is homogeneous, without loss of generality we may consider only the set

$$\mathcal{S} = \{y : y(t) \text{ is a positive solution of (1) for large } t\},$$

which is in accordance with positivity of regularly varying functions. We use the notation $\mathcal{S}^{(*)}$ when we want to emphasize that the set $\mathcal{S}$ is associated to a particular equation $(*)$. Because of the sign conditions posed on the coefficients in the statements dealing with asymptotic formulas (p is assumed to be eventually positive or eventually negative there), all solutions in $\mathcal{S}$ are eventually monotone, thus any such solution belongs to one of the classes

$$\mathcal{IS} = \{y \in \mathcal{S} : y'(t) > 0 \text{ for large } t\},\quad \mathcal{DS} = \{y \in \mathcal{S} : y'(t) < 0 \text{ for large } t\}.$$

The classes $\mathcal{IS}, \mathcal{DS}$ can further be divided into four mutually disjoint subclasses

$$\mathcal{IS}_B = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = M_y \in (0,\infty)\right\},\quad \mathcal{IS}_\infty = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = \infty\right\},$$

$$\mathcal{DS}_B = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = M_y \in (0,\infty)\right\},\quad \mathcal{DS}_0 = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = 0\right\}.$$

The so-called quasiderivative $y^{[1]}$ of $y \in \mathcal{S}$ is defined by $y^{[1]} = r\Phi(y')$. We introduce the following convention that is pertinent to the limits of solutions and their quasiderivatives:

$$\mathcal{IS}_{uv} = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} y^{[1]}(t) = v\right\},$$

$$\mathcal{DS}_{uv} = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} |y^{[1]}(t)| = v\right\};$$

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for the subscripts of $\mathcal{IS}$ and $\mathcal{DS}$, by $u = B$ and $v = B$, we mean that the value of u and v, respectively, is a positive number. Denote

$$J_p = \int_a^\infty |p(s)|\,\mathrm{d}s, \quad J_r = \int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s.$$

Recall that β is the conjugate number to α, that is, $1/\alpha + 1/\beta = 1$. The next simple lemma gathers the very basic facts about the above defined classes. For more information in this direction, see [2–4, 7].

Lemma 1 [3, 4, 7, chapters 1, 4]. *Let $p < 0$. Then,*

$$\mathcal{S} = \mathcal{IS} \cup \mathcal{DS}, \text{ where } \mathcal{IS} \neq \emptyset \neq \mathcal{DS}.$$

Moreover,

$$\mathcal{IS} = \mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty B} \cup \mathcal{IS}_{B\infty} \cup \mathcal{IS}_{BB},$$
$$\mathcal{DS} = \mathcal{DS}_{00} \cup \mathcal{DS}_{0B} \cup \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB}.$$

Let $p > 0$. If $J_r = \infty$, then $\mathcal{DS} = \emptyset$ while if $J_p = \infty$, then $\mathcal{IS} = \emptyset$. If $J_r = \infty = J_p$, then (1) is oscillatory and $\mathcal{S} = \emptyset$. If $J_r = \infty$ (and $\mathcal{S} \neq \emptyset$), then

$$\mathcal{S} = \mathcal{IS} = \mathcal{IS}_{\infty B} \cup \mathcal{IS}_{\infty 0} \cup \mathcal{IS}_{B0},$$

while if $J_p = \infty$ (and $\mathcal{S} \neq \emptyset$), then

$$\mathcal{S} = \mathcal{DS} = \mathcal{DS}_{B\infty} \cup \mathcal{DS}_{0\infty} \cup \mathcal{DS}_{0B}. \tag{8}$$

The solutions in $\mathcal{IS}_{\infty\infty}$ are called *strongly increasing* and the solutions in $\mathcal{DS}_{00}$ are called *strongly decreasing*, together they form *extremal* solutions. The solutions in $\mathcal{IS}_{\infty B}$ are called *regularly increasing* and the solutions in $\mathcal{DS}_{0B}$ are called *regularly decreasing*. The solutions in $\mathcal{IS}_{\infty B}$ and $\mathcal{DS}_{B\infty}$ are called *dominant*, the solutions in $\mathcal{IS}_{\infty 0}$ and $\mathcal{DS}_{0\infty}$ are called *intermediate*, the solutions in $\mathcal{IS}_{B0}$ and $\mathcal{DS}_{0B}$ are called *subdominant*. We stress that our results include among others precise asymptotic description of intermediate and extremal solutions, which typically belong to the "difficult" classes.

Finally note that the existence and uniqueness for the initial value problem (IVP) involving (1) is guaranteed globally as in the linear case, see [7, chapter 1].

4 | GENERAL RESULTS

The structure of the solution space and asymptotic behavior of solutions of (1) are strongly affected by the behavior of $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s$. Accordingly, we consider and introduce the following conditions and convention:

$$\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty, \quad R_D(t) := \int_a^t r^{1-\beta}(s)\,\mathrm{d}s, \tag{9}$$

$$\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty, \quad R_C(t) := \int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s. \tag{10}$$

Set

$$\omega = -\alpha, Q = R_D, \text{ and } \sigma = 1 \text{provided (9) holds}, \tag{11}$$

while

$$\omega = \alpha - 2, Q = R_C, \text{ and } \sigma = -1 \text{provided (10) holds}. \tag{12}$$

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We start with the condition guaranteeing (generalized) regular variation of all positive solutions. Because of homogeneity of the solution space, actually all nontrivial solutions are involved. Necessity of this condition is discussed in Remark 3.

Theorem 1. *Let*

$$\lim_{t\to\infty} Q^{\alpha}(t)p(t)r^{\beta-1}(t) = C \in (-\infty, K], \quad K := \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}. \tag{13}$$

We assume, in addition, nonoscillation of (1) when $C = K$ with $Q^{\alpha}(t)p(t)r^{\beta-1}(t) \not\leq K$ (in all other cases, nonoscillation is automatically guaranteed). Then,

$\mathcal{S} \subset \bigcup_{i=\{1,2\}} \mathcal{NRV}_{Q^{\sigma}}(\Phi^{-1}(\lambda_i))$ with $\mathcal{S} \cap \mathcal{NRV}_{Q^{\sigma}}(\Phi^{-1}(\lambda_i)) \neq \emptyset$, $i = 1, 2$,
where $\lambda_1 \leq \lambda_2$ are the roots of

$$|\lambda|^{\beta} - \sigma\lambda + \frac{C}{\alpha-1} = 0. \tag{14}$$

Moreover, if $p > 0$, then $\mathcal{S} = \mathcal{IS}$ when (9) holds while $\mathcal{S} = \mathcal{DS}$ when (10) holds. If $p < 0$, then $\emptyset \neq \mathcal{DS} \subset \mathcal{NRV}_{Q^{\sigma}}(\Phi^{-1}(\lambda_1))$ and $\emptyset \neq \mathcal{IS} \subset \mathcal{NRV}_{Q^{\sigma}}(\Phi^{-1}(\lambda_2))$.

Note that the properties of the roots of (14) are summarized in Lemma 2. It is easy to see that the indices of regular variation $\vartheta_i := \Phi^{-1}(\lambda_i)$, $i = 1, 2$, from the previous theorem satisfy the equation $|\vartheta|^{\alpha} - \sigma\Phi(\vartheta) + C/(\alpha-1) = 0$. Moreover, (14) has real roots only when $C \leq K$.

Remark 1. Under the setting that we consider in the general theorems on asymptotic formulas (in particular, condition (15)), condition (13) becomes necessary for the existence of a regularly varying (in the generalized sense) solution to (1). More precisely, the following statement holds: Assume that (15) is fulfilled with $\eta \neq -1$. If $\mathcal{S} \cap \mathcal{NRV}_{Q^{\sigma}}(\vartheta) \neq \emptyset$, where $\lambda = \Phi(\vartheta)$ is a real root of (14), then (13) holds and $\eta \leq \omega$. Note that in the general case (i.e., when (15) is dropped), necessary conditions have more general integral form. Such integral conditions are established in [13] and we present them in the proof of our Lemma 9, see (94) and (95), which plays an important role in the proof of this remark.

The next theorem deals with generalized slowly varying solutions. In Remark 2, we discuss optimality and connaturality of the assumptions of Theorem 2. Note that thanks to (16), Theorem 1 guarantees nonemptiness of $\mathcal{S} \cap \mathcal{NSV}_{Q^{\sigma}}$.

Theorem 2. *Assume that*

$$Q^{\sigma\omega+\alpha}|p|r^{\beta-1} \in \mathcal{RV}_{Q^{\sigma}}(\eta), \tag{15}$$

where $\omega \in \{-\alpha, \alpha-2\}$ and either $\eta \leq \omega < -1$ or $-1 < \eta \leq \omega$. If $\eta = \omega$, let, in addition,

$$\lim_{t\to\infty} Q^{\alpha}(t)p(t)r^{\beta-1}(t) = 0. \tag{16}$$

(Note that condition (16) is automatically satisfied when $\eta < \omega$.) If $y \in \mathcal{S} \cap \mathcal{NSV}_{Q^{\sigma}}$, then $y \operatorname{sgn} y' \in \Pi_{Q^{\sigma}}(Qr^{\beta-1}|y'|)$ provided $\eta = \omega$. For any $y \in \mathcal{S} \cap \mathcal{NSV}_{Q^{\sigma}}$, the following holds:

(i) If $\int_a^{\infty} (Q(s)|p(s)|r^{\beta-2}(s))^{\beta-1}\,\mathrm{d}s = \infty$ (note that this can happen only when $\eta = \omega$), then

$$y(t) = \exp\left\{-\int_a^t (1+o(1))\Phi^{-1}\left(\frac{Q(s)p(s)r^{\beta-2}(s)}{\omega+1}\right)\mathrm{d}s\right\} \tag{17}$$

as $t \to \infty$. If (9) holds, then, as $t \to \infty$,

$$\begin{aligned} &y(t) \nearrow \infty,\ y^{[1]}(t) \searrow 0 \text{ (i.e., } y \in \mathcal{IS}_{\infty 0}) \text{ when } p > 0, \\ &\text{while } y(t) \searrow 0,\ y^{[1]}(t) \nearrow 0 \text{ (i.e., } y \in \mathcal{DS}_{00}) \text{ when } p < 0. \end{aligned} \tag{18}$$

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If (10) holds, then, as $t \to \infty$,

$$\begin{aligned} &y(t) \searrow 0,\ y^{[1]}(t) \searrow -\infty \text{ (i.e., } y \in \mathcal{DS}_{0\infty}) \text{ when } p > 0, \\ &\text{while } y(t) \nearrow \infty,\ y^{[1]}(t) \nearrow \infty \text{ (i.e., } y \in \mathcal{IS}_{\infty\infty}) \text{ when } p < 0. \end{aligned} \tag{19}$$

(ii) If $\int_a^\infty (Q(s)|p(s)|r^{\beta-2}(s))^{\beta-1}\,\mathrm{d}s < \infty$, *then*

$$y(t) = M_y \exp\left\{\int_t^\infty (1+o(1))\Phi^{-1}\left(\frac{Q(s)p(s)r^{\beta-2}(s)}{\eta+1}\right)\mathrm{d}s\right\} \tag{20}$$

as $t \to \infty$, *where* $M_y = \lim_{t\to\infty} y(t) \in (0,\infty)$. *The same conclusions as in (18) and (19) hold provided we replace* $y(t) \to 0$ *and* $y(t) \to \infty$ *by* $y(t) \to M_y$. *Moreover,* $|y - M_y| \in \mathcal{SV}_{Q^\sigma}$ *and*

$$(Q^\alpha(t)|p(t)|r^{\beta-1}(t))^{\beta-1} = o(|y(t) - M_y|) \tag{21}$$

as $t \to \infty$ *when* $\eta = \omega$, *while*

$$|y(t) - M_y| \sim \frac{M_y}{|\eta+1|^{\beta-1}(\beta-1)(\omega-\eta)}\left(Q^{(\eta-\omega)(\sigma-1)+\alpha}(t)|p(t)|r^{\beta-1}(t)\right)^{\beta-1} \tag{22}$$

as $t \to \infty$ *when* $\eta < \omega$.

Remark 2. The particular values of ω (see (11), (12)), which are considered in Theorem 2, are not restrictive, since they are the results of suitable transformations applied to the equation under a general setting. Further, in that statement, we assume $\eta \le \omega$, which is also natural and not restrictive. Indeed, condition (15) can be rewritten as $Q^\alpha|p|r^{\beta-1} \in \mathcal{RV}_{Q^\sigma}(\eta-\omega)$. Hence, if $\eta > \omega$, then $\lim_{t\to\infty} Q^\alpha(t)p(t)r^{\beta-1}(t) = \infty$ by Proposition 3. According to Remark 3, we would get that there is no $\mathcal{RV}_{Q^\sigma}$ solution of (1). The setting of the theorem excludes also the case $\eta < -1 \le \omega$, which is covered in Theorem 3. The cases where the strict inequalities are replaced by possible equalities are discussed later.

The next theorem is the complement of the previous one in the sense that $\eta < -1 \le \omega$. Note that under this setting, the solution space of (1) has a quite specific structure.

Theorem 3. *Let (10) hold. Assume that* $R_C|p|r^{\beta-1} \in \mathcal{RV}_{1/R_C}(\eta)$, *where* $\eta < -1$. *Then,* $\mathcal{S} \cap \mathcal{NSV}_{1/R_C} \ne \emptyset$. *Let* $y \in \mathcal{S} \cap \mathcal{NSV}_{1/R_C}$. *Then,*

$$y \in \mathcal{IS}_{BB} \cup \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB} \text{ when } p < 0, \text{ while } y \in \mathcal{IS}_{B0} \cup \mathcal{IS}_{BB} \cup \mathcal{DS}_{BB} \text{ when } p > 0, \tag{23}$$

and all these subclasses are nonempty. If $y \in \mathcal{IS}_{B0}$ *or* $y \in \mathcal{DS}_{B0}$, *then it obeys formula (20). If* $y \in \mathcal{IS}_{BB}$ *or* $y \in \mathcal{DS}_{BB}$, *then*

$$|M_y - y(t)| \sim |N_y|^{\beta-1}R_c(t) \quad \text{and} \quad \left|N_y - y^{[1]}(t)\right| \sim \frac{M_y^{\alpha-1}}{|\eta+1|}R_C(t)|p(t)|r^{\beta-1}(t) \tag{24}$$

as $t \to \infty$, *where* $M_y = \lim_{t\to\infty} y(t) \in (0,\infty)$ *and* $N_y = \lim_{t\to\infty} y^{[1]}(t) \in \mathbb{R}\setminus\{0\}$.

Next, we examine the case where the indices of regular variation (thus the corresponding roots of (14)) are nonzero. Both eventualities, $C \ne 0$ and $C = 0$, are considered. Note that the case $C = 0$ is a complement of Theorem 2 since one root is zero and one nonzero. Moreover, with $C = 0$, the condition $|G| \in \mathcal{RV}_{Q^\sigma}(\tau)$ reduces to $Q^\alpha|p|r^{\beta-1} \in \mathcal{RV}_{Q^\sigma}(\tau)$, which is nothing but (15) where $\tau = \eta - \omega$. The lower bound $a+1$ instead of a in the integral in (25) is just the technical concern in order to avoid $Q(a) = 0$ in the denominator of the integrand; in fact, $a+1$ can be replaced by any number greater than a.

Theorem 4. *Let* $G(t) \to 0$ *as* $t \to \infty$, *where* $G := Q^\alpha pr^{\beta-1} - C$, $C \in (-\infty, K)$. *Assume that* $|G| \in \mathcal{RV}_{Q^\sigma}(\tau)$ *with some* $\tau \le 0$, *and* $\lambda_1 < \lambda_2$ *are the roots of (14); if* $C = 0$, *we consider only the nonzero root. Set* $\vartheta_i = \Phi^{-1}(\lambda_i)$, $i = 1,2$. *Let* $y \in \mathcal{S} \cap \mathcal{NRV}_{Q^\sigma}(\vartheta_i)$. *Then,* $L_{y_i}^{[Q^\sigma]}\operatorname{sgn}(L_{y_i}^{[Q^\sigma]})' \in \Pi_{Q^\sigma}(r^{\beta-1}Q(L_{y_i}^{[Q^\sigma]})')$ *provided* $\tau = 0$. *For any* $y \in \mathcal{S} \cap \mathcal{NRV}_{Q^\sigma}(\vartheta_i)$, *the following holds:*

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(i) *If* $\int_{a+1}^{\infty} |G(t)|r^{1-\beta}(t)/Q(t)\,\mathrm{d}t = \infty$ *(note that this can happen only when* $\tau = 0$*), then*

$$y_i(t) = Q^{\sigma\vartheta_i}(t)\exp\left\{-\int_{a+1}^{t}\frac{(1+o(1))r^{1-\beta}(s)G(s)}{(\alpha-1)(\Phi(\vartheta_i)-C/\vartheta_i)Q(s)}\,\mathrm{d}s\right\} \tag{25}$$

as $t\to\infty$*. If* $G>0$*, then,* $L_{y_1}^{[Q^\sigma]}(t)\nearrow\infty$ *and* $L_{y_2}^{[Q^\sigma]}(t)\searrow 0$*. If* $G<0$*, then,* $L_{y_1}^{[Q^\sigma]}(t)\searrow 0$ *and* $L_{y_2}^{[Q^\sigma]}(t)\nearrow\infty$*. If* $C\neq 0$*, then* $y_i^{[1]}(t)\searrow 0$ *(thus* $y_i\in\mathcal{IS}_{\infty 0}$*),* $i=1,2$*, when* $p>0$ *and (9) holds, while* $y_i^{[1]}(t)\searrow-\infty$ *(thus,* $y_i\in\mathcal{DS}_{0,\infty}$*),* $i=1,2$*, when* $p>0$ *and (10) holds, and* $y_1^{[1]}(t)\nearrow 0$ *(thus,* $y_1\in\mathcal{DS}_{00}$*),* $y_2^{[1]}(t)\nearrow\infty$ *(thus,* $y_2\in\mathcal{IS}_{\infty\infty}$*) when* $p<0$*. If* $C=0$ *and (9) holds, then* $y_2^{[1]}(t)\searrow 0$ *(thus,* $y_2\in\mathcal{IS}_{\infty 0}$*) when* $p>0$*, while* $y_2^{[1]}(t)\nearrow\infty$ *(thus,* $y_2\in\mathcal{IS}_{\infty\infty}$*) when* $p<0$*. If* $C=0$ *and (10) holds, then* $y_1^{[1]}(t)\searrow-\infty$ *(thus,* $y_1\in\mathcal{DS}_{0,\infty}$*) when* $p>0$*, while* $y_1^{[1]}(t)\nearrow 0$ *(thus,* $y_1\in\mathcal{DS}_{00}$*) when* $p<0$*.*

(ii) *If* $\int_{a+1}^{\infty} |G(t)|r^{1-\beta}(t)/Q(t)\,\mathrm{d}t < \infty$*, then*

$$y_i(t) = M_iQ^{\sigma\vartheta_i}(t)\exp\left\{\int_{t}^{\infty}\frac{(1+o(1))r^{1-\beta}(s)G(s)}{(\alpha-1)(\Phi(\vartheta_i)-C/\vartheta_i+\tau|\vartheta_i|^{\alpha-2})Q(s)}\,\mathrm{d}s\right\} \tag{26}$$

as $t\to\infty$*, where* $M_i=\lim_{t\to\infty}L_{y_i}^{[Q^\sigma]}(t)\in(0,\infty)$*,* $i=1,2$*. As for monotonicity of* $y_iQ^{-\sigma\vartheta_i}$*, the same conclusions as in (i) hold with the modification that the limit values are replaced by* M_i*. As for the asymptotic behavior of* $y_i^{[1]}$*, it is the same as in (i) when* $C\neq 0$*. If* $C=0$*, then for the monotonicity of* $y_1^{[1]}$ *and* $y_2^{[1]}$ *the same conclusions as in (i) hold with the modification that the limit values are replaced by a positive number. Moreover, if* $\tau=0$*, then* $|M_i-L_{y_i}^{[Q^\sigma]}|\in\mathcal{SV}_{Q^\sigma}$ *and*

$$|G(t)| = o(|M_i - L_{y_i}^{[Q^\sigma]}(t)|) \tag{27}$$

as $t\to\infty$*,* $i=1,2$*, while if* $\tau<0$*, then*

$$\left|M_i - L_{y_i}^{[Q^\sigma]}(t)\right| \sim \frac{M_i|G(t)|}{|\tau|(\alpha-1)|\Phi(\vartheta_i)-C/\vartheta_i+\tau|\vartheta_i|^{\alpha-2}|} \tag{28}$$

as $t\to\infty$*,* $i=1,2$*.*

5 | EQUATIONS WITH REGULARLY VARYING COEFFICIENTS

If the coefficients of Equation (1) are regularly varying, then both the assumptions and the formulas of the general statements from the previous section take a quite lucid form. As we will see later, the next statement is a corollary of Theorem 1 when $r\in\mathcal{RV}(\gamma)$ with $\gamma\neq\alpha-1$. It however holds true also for $\gamma=\alpha-1$. Properties of the roots of Equation (30) are summarized in Lemma 2.

Theorem 5. *Let* $r\in\mathcal{RV}(\gamma)$*,* $\gamma\in\mathbb{R}$*, and*

$$\lim_{t\to\infty}\frac{t^\alpha p(t)}{r(t)} = C_\gamma\in(-\infty,K_\gamma],\quad K_\gamma=\left(\frac{|\alpha-1-\gamma|}{\alpha}\right)^\alpha. \tag{29}$$

We assume, in addition, nonoscillation of (1) when $C_\gamma=K_\gamma$ *with* $t^\alpha p(t)/r(t)\not\leq K_\gamma$ *(in all other cases, nonoscillation is automatically guaranteed). Then,* $\mathcal{S}\subset\bigcup_{i=1,2}\mathcal{NRV}(\Phi^{-1}(\eta_i))$ *with* $\mathcal{S}\cap\mathcal{NRV}(\Phi^{-1}(\eta_i))\neq\emptyset$*,* $i=1,2$*, where* $\eta_1\leq\eta_2$ *are the roots of*

$$F_\gamma(\eta) := |\eta|^\beta - \frac{\alpha-1-\gamma}{\alpha-1}\eta + \frac{C_\gamma}{\alpha-1} = 0. \tag{30}$$

Remark 3.

(i) In view of below presented Lemma 9, if $|p|$ and r regularly varying (and that is what we assume there to establish asymptotic formulas), (sufficient) condition (29) becomes necessary for the existence of a regularly varying solution.

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More precisely, we assume there $|p| \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\gamma)$ with $\delta \neq -1, \gamma \neq \alpha - 1$ (the equalities lead to the critical case, which is discussed later). In addition, we obtain that necessarily $\delta + \alpha \leq \gamma$, thus assuming this relation between the indices of regular variation of r and p in our statements is natural and means no restriction.

(ii) Let $|p| \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\gamma)$ with $\gamma \neq \alpha - 1$. According to [32, Lemma 10], if y is a slowly varying solution of (1), which tends to 0 or ∞, then $\gamma = \delta + \alpha$. Moreover, $y^{[1]}$ cannot tend to a nonzero real constant.

The next statement is a corollary of Theorem 2 for p and r regularly varying. As already noted, the inequality $\delta + \alpha \leq \gamma$ is not restrictive, since if $\delta + \alpha > \gamma$, then there are no regularly varying solutions.

Corollary 1. *Let $|p| \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\gamma)$, $\delta + \alpha \leq \gamma$, with $\delta < -1$ and $\gamma < \alpha - 1$ or $\delta > -1$ and $\gamma > \alpha - 1$. Assume that $\lim_{t\to\infty} t^\alpha p(t)/r(t) = 0$ (note that this condition is automatically satisfied when $\delta + \alpha < \gamma$). Let $y \in \mathcal{S} \cap \mathcal{NSV}$ (note that the set is nonempty).*

(i) If $\int_a^\infty |sp(s)/r(s)|^{\beta-1}\,\mathrm{d}s = \infty$ (note that this can happen only when $\delta + \alpha = \gamma$), then

$$y(t) = \exp\left\{-\int_a^t (1 + o(1))\Phi^{-1}\left(\frac{L_p(s)}{(\delta+1)L_r(s)}\right)\frac{1}{s}\,\mathrm{d}s\right\} \tag{31}$$

as $t \to \infty$.

(ii) If $\int_a^\infty |sp(s)/r(s)|^{\beta-1}\,\mathrm{d}s < \infty$, then

$$y(t) = M_y \exp\left\{\int_t^\infty (1 + o(1))\Phi^{-1}\left(\frac{L_p(s)}{(\delta+1)L_r(s)}\right)s^{(\delta+1-\gamma)(\beta-1)}\,\mathrm{d}s\right\} \tag{32}$$

as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t) \in (0, \infty)$. Moreover, if $\delta + \gamma = \alpha$, then $|y - M_y| \in \mathcal{SV}$ and

$$\left|\frac{L_p(t)}{L_r(t)}\right|^{\beta-1} = o(|y(t) - M_y|) \tag{33}$$

as $t \to \infty$, while if $\delta + \alpha < \gamma$, then

$$|M_y - y(t)| \sim \frac{M_y(\alpha - 1)}{|\delta + 1|^{\beta-1}(\gamma - \delta - \alpha)}\left|\frac{t^\alpha p(t)}{r(t)}\right|^{\beta-1} \tag{34}$$

as $t \to \infty$.

In Corollary 1, we assume $\delta < -1$ and $\gamma < \alpha - 1$ or $\delta > -1$ and $\gamma > \alpha - 1$; note that if $\delta + \alpha = \gamma$, then these are the only combinations of strict inequalities that can occur. This suggests the problem of tackling the other combinations. If $\delta > -1$ and $\gamma < \alpha - 1$, then there are no regularly varying solutions of (1). Indeed, if $p < 0$, then, on one hand, (1) is nonoscillatory by the Sturm-type comparison theorem [7, Chapter 1], on the other hand, all its eventually positive solutions belong to the de Haan classes Γ and Γ_- (see [33]), which form subsets of rapidly varying solutions, and thus cannot be regularly varying. If $p > 0$, then we have, by the Karamata integration theorem (Theorem 7), $\int_a^\infty p(s)\,\mathrm{d}s = \infty = \int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s$, and, in view of the Hille–Wintner–type criterion [7, Chapter 1], Equation (1) is nonoscillatory, so there is even no eventually positive solution. The next statement deals with the remaining case $\delta < -1$ and $\gamma > \alpha - 1$; note that this can occur only when $\delta + \alpha < \gamma$. The cases where the strict inequality is replaced by the equality is discussed later.

Corollary 2. *Let $|p| \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\gamma)$ with $\delta < -1$ and $\gamma > \alpha - 1$. Assume $y \in \mathcal{S} \cap \mathcal{NSV}$ (note that the set is nonempty). Then, (23) holds and all the subclasses in (23) are nonempty. If $y \in \mathcal{IS}_{B0}$ or $y \in \mathcal{DS}_{B0}$, then it obeys formula (32). If $y \in \mathcal{IS}_{BB}$ or $y \in \mathcal{DS}_{BB}$, then*

$$|M_y - y(t)| \sim \frac{|N_y|^{\beta-1}}{(\beta-1)(\gamma-\alpha+1)}tr^{1-\beta}(t) \quad \text{and} \quad \left|N_y - y^{[1]}(t)\right| \sim \frac{M_y^{\alpha-1}}{(\beta-1)(\delta+1)}|p(t)|r^{\beta-1}(t) \tag{35}$$

as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t) \in (0, \infty)$ and $N_y = \lim_{t\to\infty} y^{[1]}(t) \in \mathbb{R} \setminus \{0\}$.

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In view of the above presented results, it is now natural to be interested in formulas in the case where the limit $\lim_{t\to\infty} t^\alpha p(t)/r(t)$ is nonzero or it is zero and we consider non-$\mathcal{SV}$ solutions. If $r \in \mathcal{RV}(\gamma)$ with $\gamma \neq \alpha - 1$ and $\lim_{t\to\infty} t^\alpha p(t)/r(t) = C_\gamma \in (0, K_\gamma)$, we can obtain—as a corollary of Theorem 4—the formulas for $y_i \in \mathcal{S} \cap \mathcal{NRV}(\Phi^{-1}(\eta_i))$, $i = 1, 2$, η_1, η_2 being the nonzero roots of (30). For illustration, let us present the case where the integral is divergent:

$$y_i(t) = t^{\Phi^{-1}(\eta_i)} L_r(t)^{\Phi^{-1}(\eta_i)/(\gamma+1-\alpha)} \exp\left\{ -\int_a^t \frac{(1+o(1))}{(\alpha-1)(\eta_i - C_\gamma/\Phi^{-1}(\eta_i))s} \left((1+o(1)) \frac{s^\alpha p(s)}{r(s)} - C_\gamma \right) \mathrm{d}s \right\}.$$

The formula in the case where the integral is convergent is more complicated. We prefer to prove directly a different variant under the assumption $r \in \mathcal{NRV}(\gamma) \cap C^1$, where γ can be arbitrary real number. In fact, for the case $C_\gamma = 0$, there is another approach how to obtain asymptotic formulas for non-$\mathcal{SV}$ solutions. It is based on transforming Equation (1) into reciprocal Equation (89) to which we apply Corollary 1, see [32] for details.

Theorem 6. *Let $r \in \mathcal{NRV}(\gamma) \cap C^1$, $\gamma \in \mathbb{R}$, and $\lim_{t\to\infty} t^\alpha p(t)/r(t) = C_\gamma \in (-\infty, K_\gamma)$. Assume that $|G_i| \in \mathcal{RV}(\tau_i)$, $\tau_i \leq 0$, where*

$$G_i(t) := \frac{t^\alpha p(t)}{r(t)} - C_\gamma + \eta_i \left(\frac{t r'(t)}{r(t)} - \gamma \right), \tag{36}$$

$i = 1, 2$, where $\eta_1 < \eta_2$ are the roots of (30); if $C_\gamma = 0$, we consider only the nonzero root. Suppose that $\gamma + \alpha(\Phi^{-1}(\eta_2) - 1) + \tau_2 > -1$ (note that if $\tau_2 = 0$, then this condition is automatically fulfilled). Let $y_i \in \mathcal{S} \cap \mathcal{NRV}(\Phi^{-1}(\eta_i))$, $i = 1, 2$.

(i) If, for $i = 1$ or $i = 2$, $\int_a^\infty |G_i(s)|/s \, \mathrm{d}s = \infty$ (note that this can happen only when $\tau_i = 0$), then

$$y_i(t) = t^{\Phi^{-1}(\eta_i)} \exp\left\{ -\int_a^t (1+o(1)) \frac{G_i(s)}{(\alpha-1)(\eta_i - C_\gamma/\Phi^{-1}(\eta_i))s} \, \mathrm{d}s \right\} \tag{37}$$

as $t \to \infty$.

(ii) If, for $i = 1$ or $i = 2$, $\int_a^\infty |G_i(s)|/s \, \mathrm{d}s < \infty$, then

$$y_i(t) = M_i t^{\Phi^{-1}(\eta_i)} \exp\left\{ \int_t^\infty (1+o(1)) \frac{G_i(s)}{(\alpha-1)(\eta_i - C_\gamma/\Phi^{-1}(\eta_i) + \tau_i |\eta_i|^{(\alpha-2)(\beta-1)})s} \, \mathrm{d}s \right\} \tag{38}$$

as $t \to \infty$, where $M_i = \lim_{t\to\infty} y_i(t)/t^{\xi_i} \in (0, \infty)$. Moreover, if $\tau_i = 0$, then $|M_i - L_{y_i}| \in \mathcal{SV}$ and $|G_i(t)| = o(|M_i - L_{y_i}(t)|)$ as $t \to \infty$, while if $\tau_i < 0$, then

$$|M_i - L_{y_i}(t)| \sim \frac{M_i |G_i(t)|}{|\tau_i|(\alpha-1)|\eta_i - C_\gamma/\Phi^{-1}(\eta_i) + \tau_i |\eta_i|^{(\alpha-2)(\beta-1)}|}$$

as $t \to \infty$.

Remark 4. A closer examination of the proof of Theorem 6 shows that for y_1, y_2 from that theorem, we have

$$\left| \Phi\left(\frac{t y_i'(t)}{y_i(t)} \right) - \eta_i \right| \sim \left| \frac{G_i(t)}{\eta_i - C_\gamma/\Phi^{-1}(\eta_i) + \tau_i |\eta_i|^{(\alpha-2)(\beta-1)}} \right|$$

as $t \to \infty$, $i = 1, 2$.

Example 1.

(i) In general, the case when $\int_a^\infty |G_k(s)|/s \, \mathrm{d}s = \infty$ and $\int_a^\infty |G_{3-k}(s)|/s \, \mathrm{d}s < \infty$ for one of $k \in \{1, 2\}$ can generally occur even with $\tau_1 = \tau_2 = 0$. Indeed, let $r(t) = t^\gamma L_r(t)$, $\gamma \in \mathbb{R}$, where $L_r \in \mathcal{NSV} \cap C^1$ with $\lim_{t\to\infty} L_r(t) = \infty$, and $p(t) = (C_\gamma L_r(t) - \eta_1 + \eta_2/L_r(t)) t^{\gamma-\alpha}$, $C_\gamma \in (-\infty, K_\gamma) \setminus \{0\}$. Then, $t^\alpha p(t)/r(t) = C_\gamma - \eta_1/L_r(t) + \eta_2/L_r^2(t) \to C_\gamma$ as $t \to \infty$.

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Since $tr'(t)/r(t) = \gamma + tL_r'(t)/L_r(t)$, we get

$$G_i(t) = -\frac{\eta_1}{L_r(t)} + \frac{\eta_2}{L_r^2(t)} + \eta_i \frac{tL_r'(t)}{L_r(t)}.$$

Choose $L_r(t) = \ln t$. Then, $G_1(t) = \eta_2/\ln^2 t$ and $G_2(t) = (\eta_2 - \eta_1)/\ln t + \eta_2/\ln^2 t \sim (\eta_2 - \eta_1)/\ln t$ as $t \to \infty$. Hence, $|G_1| \in \mathcal{SV}$ and $G_2 \in \mathcal{SV}$. Moreover, $\int_2^\infty |G_1(t)|/t\,\mathrm{d}t < \infty$ and $\int_2^\infty G_2(t)/t\,\mathrm{d}t = \infty$.

(ii) If $\eta_2 < 0$ and $L_r = \ln$ in part (i), then $G_1(t) < 0$ and $G_2(t) > 0$ for large t, thus G_1 and G_2 can have different signs

(iii) If $r(t) = t^\gamma$, then $G_1(t) = G_2(t) = t^\alpha p(t)/r(t) - C_\gamma$.

6 | EQUATIONS WITH PERTURBED EXPONENTIAL COEFFICIENTS

We are able to use the results to obtain asymptotic formulas also for some equations with non-RV coefficients, which can take, for example, the form of exponentials perturbed by RV functions (thus the perturbations "vary slower" than exponentials). One could directly apply the general results involving generalized RV functions (which allow exponentials). Nevertheless, next we offer a slightly modified approach; the proof is based on applying a suitable transformation and utilizing existing results.

Corollary 3. *Let $p(t) = \mathrm{e}^{\xi t} g(t)$ and $r(t) = \mathrm{e}^{\mu t} h(t)$, where $\xi, \mu \in \mathbb{R}$, $g \in C[a,\infty)$, and $h \circ \ln \in \mathcal{NSV} \cap C^1[a,\infty)$. Assume that $\lim_{t\to\infty} \mathrm{e}^{t(\xi-\mu)} g(t)/h(t) = C \in (-\infty, (|\mu|/\alpha)^\alpha)$ and*

$$|K_i \circ \ln| \in \mathcal{RV}(\tau_i), \quad K_i(t) := \frac{\mathrm{e}^{t(\xi-\mu)} g(t)}{h(t)} - C + \eta_i \frac{h'(t)}{h(t)},$$

$i = 1, 2$, where $\tau_i \le 0$ and $\eta_1 \le \eta_2$ are the roots of the equation

$$|\eta|^\beta + \mu/(\alpha - 1) + C/(\alpha - 1) = 0. \tag{39}$$

Let $\mu + \Phi^{-1}(\eta_2) + \tau_2 > 0$ (note that this condition is automatically fulfilled when $\tau_2 = 0$). If $C = 0$ and $\eta_k = 0$ for $k = 1$ or $k = 2$, we assume, in addition, $\mu < 0$ or $\tau_k + \mu > 0$, and the condition on normality and C^1 of $h \circ \ln$ can be dropped. Then, $S \subset \bigcup_{i\in\{1,2\}} \mathcal{NRV}_{\exp}(\Phi^{-1}(\eta_i))$. Moreover, for any $y \in S \cap \mathcal{NRV}(\Phi^{-1}(\eta_i))$, the following holds:

(i) If η_i is a nonzero root and $\int_a^\infty |K_i(s)|\,\mathrm{d}s = \infty$, then

$$y_i(t) = \exp\left\{ \Phi^{-1}(\eta_i) - \int_a^t (1 + o(1)) \frac{K_i(s)}{(\alpha - 1)(\eta_i - C/\Phi^{-1}(\eta_i))}\,\mathrm{d}s \right\} \tag{40}$$

as $t \to \infty$.

(ii) If η_i is a nonzero root and $\int_a^\infty |K_i(s)|\,\mathrm{d}s < \infty$, then

$$y_i(t) = M_i \exp\left\{ \Phi^{-1}(\eta_i) + \int_t^\infty (1 + o(1)) \frac{K_i(s)}{(\alpha - 1)(\eta_i - C/\Phi^{-1}(\eta_i))}\,\mathrm{d}s \right\} \tag{41}$$

as $t \to \infty$, where $M_i = \lim_{t\to\infty} y_i(t)$.

(iii) If $\eta_i = 0$ and $\int_a^\infty |s^{\xi-\mu} g(s)/h(s)|^{\beta-1}\,\mathrm{d}s = \infty$, then

$$y_i(t) = \exp\left\{ -\int_a^t (1 + o(1)) \Phi\left(\frac{s^{\xi-\mu} g(s)}{(\tau_i + \mu) h(s)} \right) \mathrm{d}s \right\} \tag{42}$$

as $t \to \infty$.

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(iv) If $\eta_i = 0$ *and* $\int_a^\infty |s^{\xi-\mu} g(s)/h(s)|^{\beta-1}\,\mathrm{d}s < \infty$, *then*

$$y_i(t) = M_i \exp\left\{\int_t^\infty (1+o(1))\Phi\left(\frac{s^{\xi-\mu}g(s)}{(\tau_i+\mu)h(s)}\right)\mathrm{d}s\right\} \tag{43}$$

as $t \to \infty$, *where* $M_i = \lim_{t\to\infty} y_i(t)$.

Remark 5. Condition $h\circ\ln \in \mathcal{NSV}$ is satisfied, in particular, when $h \in C^1 \cap \bigcup_{\vartheta\in\mathbb{R}} \mathcal{NRV}(\vartheta)$. If we assume, in addition, $g \in C \cap \bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}(\vartheta)$, then the coefficients in (1) can be seen as exponentials, which are perturbed by regularly varying functions. Note that the condition $h \in \bigcup_{\vartheta\in\mathbb{R}} \mathcal{NRV}(\vartheta)$ can be generalized to $h \in \mathcal{NRB} \cap C^1$. Indeed, from representation (7), we get that $-\infty < \liminf_{t\to\infty} th'(t)/h(t) \le \limsup_{t\to\infty} th'(t)/h(t) < \infty$. Consequently,

$$\frac{s\frac{\mathrm{d}(h\circ\ln)}{\mathrm{d}s}(s)}{(h\circ\ln)(s)} = \frac{h'(\ln s)}{h(\ln s)} = \frac{h'(\ln s)\ln s}{h(\ln s)}\cdot\frac{1}{\ln s} \to 0$$

as $s \to \infty$. If $\xi = \eta \neq 0$, then it is sufficient to take $|g(t)| \sim Ch(t)$ with $C \neq 0$ or $|g(t)| = o(h(t))$ as $t \to \infty$.

7 | CRITICAL SETTING

Looking at Corollary 1 and Theorem 6, we see that the cases when $\delta = -1$ and $C_\gamma = K_\gamma$ are not covered. These critical settings (which lead to double root case in Theorem 5) are somehow delicate and require a different approach. Nevertheless, the next statements show that sophisticated utilization of Theorem 2 and Theorem 4 enables us to include also these settings. First, we deal with the case when $\delta = -1$ and $C_\gamma = 0$. We prefer to choose somehow specific setting which however serves as an important prototype and allow us to establish precise formulas in a nice form.

Corollary 4. *Let* $p(t) = \frac{1}{t} f_p(t)\ln^\mu t$ *and* $r(t) = t^{\alpha-1}\ln^\zeta t$, *where* $\mu+\alpha-\zeta \le 0$, $f_p = g_p\circ\ln$ *with* $|g_p| \in \mathcal{SV}\cap C$, *and either* $\mu < -1$ *and* $\zeta < \alpha-1$ *or* $\mu > -1$ *and* $\zeta > \alpha-1$. *If* $\mu+\alpha = \zeta$, *assume, in addition,* $\lim_{t\to\infty} f_p(t) = 0$. *If* $f_p > 0$, *assume, in addition, nonoscillation of (1). Then,*

$$\mathcal{S} \subset \mathcal{NSV}_{Q^\sigma} \cup \mathcal{NRV}_{Q^\sigma}(\sigma) \subset \mathcal{NSV}, \tag{44}$$

where $\sigma = \mathrm{sgn}(\alpha-1-\zeta)$ *and* $Q(t) = \ln^\Theta t/|\Theta|$, $\Theta := \zeta(1-\beta)+1$. *Moreover, the following holds:*

(i-a) If $y \in \mathcal{S}\cap\mathcal{NSV}_{Q^\sigma}$, $\mu+\alpha = \zeta$, *and* $\int_e^\infty |f_p(s)|^{\beta-1}/(s\ln s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = \exp\left\{-\int_e^t (1+o(1))\frac{1}{s\ln s}\Phi^{-1}\left(\frac{f_p(s)}{\mu+1}\right)\mathrm{d}s\right\} \tag{45}$$

as $t \to \infty$.

(i-b) If $y \in \mathcal{S}\cap\mathcal{NSV}_{Q^\sigma}$ *and* $\int_e^\infty (\ln s)^{(\beta-1)(\mu+\alpha-\zeta)-1}|f_p(s)|^{\beta-1}/s\,\mathrm{d}s < \infty$, *then*

$$y(t) = M_y \exp\left\{\int_t^\infty (1+o(1))\frac{(\ln s)^{(\beta-1)(\mu+\alpha-\zeta)-1}}{s}\Phi^{-1}\left(\frac{f_p(s)}{\mu+1}\right)\mathrm{d}s\right\} \tag{46}$$

as $t \to \infty$, *where* $M_y = \lim_{t\to\infty} y(t) \in (0,\infty)$.

(ii-a) If $y \in \mathcal{S}\cap\mathcal{NRV}_{Q^\sigma}(\sigma)$ *and* $\int_e^\infty |f_p(s)|/(s\ln s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = \ln^\Theta t \exp\left\{\int_e^t \frac{(1+o(1))f_p(s)}{\Phi(\Theta)(\alpha-1)s\ln s}\,\mathrm{d}s\right\} \tag{47}$$

as $t \to \infty$.

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(ii-b) If $y \in \mathcal{S} \cap \mathcal{NRV}_{Q^\sigma}(\sigma)$ *and* $\int_e^\infty (\ln s)^{\mu+\alpha-\zeta-1}|f_p(s)|/s\,\mathrm{d}s < \infty$, *then*

$$y(t) = N_y \ln^\Theta t \exp\left\{-\int_t^\infty \frac{(1+o(1))(\ln s)^{\mu+\alpha-\zeta-1} f_p(s)}{\Phi(\Theta)(\alpha-1)s}\,\mathrm{d}s\right\} \tag{48}$$

as $t \to \infty$, *where* $N_y = \lim_{t\to\infty} |\Theta| y(t)/\ln^\Theta t \in (0,\infty)$

The condition $\lim_{t\to\infty} f_p(t) = 0$ in Corollary 4 suggests the natural problem to be solved, namely, the setting $\lim_{t\to\infty} f_p(t) = C$ where $C \neq 0$. We will not go into details; note just that this case can be studied via arguments similar to those in the proof of the previous corollary with applying Theorem 4. Note that the critical cases (no matter whether C is zero or not) can be treated also by a slightly different approach, namely, in the first step we use logarithmic transformation and then we apply to the transformed equation the statements where regular variation of the coefficients is assumed (Theorem 5, Corollary 1, and Theorem 6).

Another direction concerns the fact that borderline condition $\mu = -1$ is not allowed in the previous statement. The next corollary deals just with this critical case. Note that we consider the coefficients in a certain special form, just to demonstrate the main ideas. However, a generalization or an extension is possible. In particular, the coefficients can further be perturbed (in such a way that their dominant behavior is not violated too much) and/or the settings can be considered where (10) holds or the limit $\lim_{t\to\infty}(\ln\ln t)^\alpha f_p(t)$ is nonzero.

Corollary 5. *Let* $p(t) = \frac{1}{t\ln t} f_p(t)$ *and* $r(t) = t^{\alpha-1}\ln^{\alpha-1} t$; $\lim_{t\to\infty}(\ln\ln t)^\alpha f_p(t) = 0$ *and* $f_p = h_p \circ \ln \circ \ln$ *with* $|h_p| \in \mathcal{RV}(-\alpha) \cap C$. *If* $f_p > 0$, *assume, in addition, nonoscillation of (1). Then,*

$$\mathcal{S} \subset \mathcal{NSV}_{\ln\circ\ln} \cup \mathcal{NRV}_{\ln\circ\ln}(1) \subset \mathcal{NSV}. \tag{49}$$

Moreover, the following holds:

(i-a) If $y \in \mathcal{S} \cap \mathcal{NSV}_{\ln\circ\ln}$ *and* $\int_e^\infty |f_p(s)\ln\ln s|^{\beta-1}/(s\ln s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = \exp\left\{\int_e^t (1+o(1))\frac{1}{s\ln s}\Phi^{-1}\left(\frac{f_p(s)\ln\ln s}{\alpha-1}\right)\mathrm{d}s\right\} \tag{50}$$

as $t \to \infty$.

(i-b) If $y \in \mathcal{S} \cap \mathcal{NSV}_{\ln\circ\ln}$ *and* $\int_e^\infty |f_p(s)\ln\ln s|^{\beta-1}/(s\ln s)\,\mathrm{d}s < \infty$, *then*

$$y(t) = M_y \exp\left\{-\int_t^\infty (1+o(1))\frac{1}{s\ln s}\Phi^{-1}\left(\frac{f_p(s)\ln\ln s}{\alpha-1}\right)\mathrm{d}s\right\} \tag{51}$$

as $t \to \infty$, *where* $M_y = \lim_{t\to\infty} y(t) \in (0,\infty)$ *and*

$$(\ln\ln t)^\alpha |f_p(t)| = o(|y(t) - M_y|^{\beta-1}) \tag{52}$$

as $t \to \infty$.

(ii-a) If $y \in \mathcal{S} \cap \mathcal{NRV}_{\ln\circ\ln}(1)$ *and* $\int_e^\infty (\ln\ln s)^\alpha |f_p(s)|^{\beta-1}/(s\ln s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = \ln\ln t \exp\left\{-\int_e^t (1+o(1))\frac{(\ln\ln s)^{\alpha-1}}{(\alpha-1)s\ln s} f_p(s)\,\mathrm{d}s\right\} \tag{53}$$

as $t \to \infty$.

(ii-b) If $y \in \mathcal{S} \cap \mathcal{NRV}_{\ln\circ\ln}(1)$ *and* $\int_e^\infty (\ln\ln s)^\alpha |f_p(s)|^{\beta-1}/(s\ln s)\,\mathrm{d}s < \infty$, *then*

$$y(t) = N_y \ln\ln t \exp\left\{\int_t^\infty (1+o(1))\frac{(\ln\ln s)^{\alpha-1}}{(\alpha-1)s\ln s} f_p(s)\,\mathrm{d}s\right\} \tag{54}$$

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as $t \to \infty$, *where* $N_y = \lim_{t\to\infty} y(t)/\ln\ln t \in (0,\infty)$ *and*

$$(\ln\ln t)^{\alpha}|f_p(t)| = o(|y(t) - N_y/\ln\ln t|^{\beta-1}) \tag{55}$$

as $t \to \infty$.

If $C_\gamma = K_\gamma$, then all eventually positive solutions of (1) are regularly varying with the index $(\alpha - 1 - \gamma)/\alpha$. In fact, Equation (1) under these conditions can be understood as the generalized Euler equation in the critical setting. Asymptotic formulas in this delicate case can be obtained by combination of the ideas from the proofs of Theorem 6, Corollary 4, and Corollary 5. Actually, one can think about further extension of these results in the sense of iterating the process, which transforms the critical setting into noncritical one. Perturbations of generalized Riemann–Weber differential equations could be treated in such a framework. Since the detailed analysis of this direction requires a lot of space, we do not include it into this paper.

8 | POINCARÉ–PERRON SOLUTIONS AND THEIR ASYMPTOTICS

Consider a half-linear equation in the form

$$(\Phi(y'))' + a(t)\Phi(y') + b(t)\Phi(y) = 0, \tag{56}$$

where a, b are continuous functions on $[t_0, \infty)$. The so-called Poincaré–Perron solutions of an equation are the solutions y satisfying $\lim_{t\to\infty} y'(t)/y(t) \in \mathbb{R}$, see [5, 26, 27] for the basic results in the linear case. Theory of half-linear equations in the framework of regular variation enables us to generalize the linear case into the half-linear case for second-order equations. In particular, we can establish conditions guaranteeing that Poincaré–Perron solutions exist and, in fact, all solutions are of this kind, see [31]. Note that they are then rapidly varying rather than regularly varying, see Remark 6-(iii). Moreover, thanks to our theory we can derive quite precise asymptotic formulas for these solutions; this is the content of the following corollary.

Corollary 6. *Let* $\lim_{t\to\infty} a(t) = A \in \mathbb{R}$ *and* $\lim_{t\to\infty} b(t) = B \in (-\infty, (|A|/\alpha)^{\alpha})$. *Assume that* $|H_i| \in \mathcal{RV}_{\exp}(\tau_i)$, $\tau_i \le 0$, *where*

$$H_i(t) := b(t) - B + \Phi(\lambda_i)(a(t) - A),$$

$\lambda_1 < \lambda_2$ *being the (real) roots of*

$$(\alpha - 1)|\lambda|^{\alpha} + A\Phi(\lambda) + B = 0. \tag{57}$$

Then

$$\mathcal{S} \subset \bigcup_{i\in\{1,2\}} \mathcal{NRV}_{\exp}(\lambda_i), \ \textit{i.e.,} \ \lim_{t\to\infty} \frac{y'(t)}{y(t)} \in \{\lambda_1, \lambda_2\} \tag{58}$$

for any $y \in \mathcal{S}$. *Moreover, the following holds:*

(i) For $y \in \mathcal{S} \cap \mathcal{NRV}_{\exp}(\lambda_i)$, $i = 1$ *or* $i = 2$, *where we assume* $\lambda_i \neq 0$, *one has the following:*
(i-a) If $\int_a^\infty |H_i(t)|\,\mathrm{d}t = \infty$ *(note that the divergence can occur only when* $\tau_i = 0$*), then*

$$y(t) = \exp\left\{\lambda_i t - \int_a^t (1 + o(1))\frac{H_i(s)}{(\alpha - 1)(\Phi(\lambda_i) - B/\lambda_i)}\,\mathrm{d}s\right\} \tag{59}$$

as $t \to \infty$.
(i-b) If $\int_a^\infty |H_i(t)|\,\mathrm{d}t < \infty$, *then*

$$y(t) = M\exp\left\{\lambda_i t + \int_t^\infty (1 + o(1))\frac{H_i(s)}{(\alpha - 1)(\Phi(\lambda_i) - B/\lambda_i + \tau_i|\lambda_i|^{\alpha-2})}\,\mathrm{d}s\right\} \tag{60}$$

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as $t \to \infty$, *where* $M = \lim_{t\to\infty} y(t)\exp\{-\lambda_i t\} \in (0,\infty)$.

(ii) For $y \in S \cap \mathcal{NSV}_{\exp}$, *under the assumption* $A < 0$ *or* $\tau + A > 0$, *where* $\tau_0 := \tau_i \le 0$ *corresponds to the zero root* λ_i *(and this can happen only when* $B = 0$*), one has the following:*

(ii-a) If $\int_a^\infty |b(t)|^{\beta-1}\,\mathrm{d}t = \infty$ *(note that the divergence can occur only when* $\tau_0 = 0$*), then*

$$y(t) = \exp\left\{-\int_a^t (1+o(1))\Phi^{-1}\left(\frac{b(s)}{A}\right)\mathrm{d}s\right\} \tag{61}$$

as $t \to \infty$.

(ii-b) If $\int_a^\infty |b(t)|^{\beta-1}\,\mathrm{d}t < \infty$, *then*

$$y(t) = M\exp\left\{\int_t^\infty (1+o(1))\Phi^{-1}\left(\frac{b(s)}{A+\tau_0}\right)\mathrm{d}s\right\} \tag{62}$$

as $t \to \infty$, *where* $M = \lim_{t\to\infty} y(t) \in (0,\infty)$.

Remark 6.

(i) Similar ideas, which are used in the proof of Corollary 6, can be employed to handle the critical case $B = (|A|/\alpha)^\alpha$; Corollary 5 would find application here.

(ii) Next we indicate how the setting in our nonlinear Poincaré–Perron theorem can be related to equation (1) where the coefficients are exponentials multiplied by perturbations which vary slower than exponentials, cf. also Section 6. Consider, for example, $p(t) = \mathrm{e}^{\xi t}g(t)$, $r(t) = \mathrm{e}^{\xi t}h(t)$, where $\xi \neq 0$, $h \in \bigcup_{\vartheta\in\mathbb{R}} \mathcal{NRV}(\vartheta)\cap C^1$, and $|g(t)| \sim |C|h(t)$ with $C \neq 0$ or $|g(t)| = o(h(t))$ as $t \to \infty$, $g \in C$. Equation (1) with such coefficients can easily be obtained from (56) as follows. Introduce the relations between a, b and g, h by

$$h(t) = \exp\left\{\int_{t_0}^t (a(s)-A)\,\mathrm{d}s\right\} \quad\text{and}\quad g(t) = b(t)\exp\left\{\int_{t_0}^t (a(s)-A)\,\mathrm{d}s\right\}.$$

Set $\xi = A$. Then $a(t) = \xi + h'(t)/h(t)$ and $b(t) = g(t)/h(t)$. Plugging these expressions into (56) and multiplying by h, we obtain

$$h(t)(\Phi(y'))' + (Ah(t) + h'(t))\Phi(y') + g(t)\Phi(y) = 0,$$

which can be rewritten as

$$(h(t)\Phi(y'))' + \xi h(t)\Phi(y') + g(t)\Phi(y) = 0.$$

Multiplying by $\mathrm{e}^{\xi t}$ we get

$$(\mathrm{e}^{\xi t}h(t)\Phi(y'))' + \mathrm{e}^{\xi t}g(t)\Phi(y) = 0,$$

which is of the form (1). Since $th'(t)/h(t) \to \vartheta$ as $t \to \infty$, we have $h'(t)/h(t) \to 0$ as $t \to \infty$, and so $a(t) \to \xi = A$ as $t \to \infty$. Moreover, $b(t) \to C$ as $t \to \infty$ where C can attain also the zero value. We thus see that this setting in equation (1) corresponds with with asymptotically constant coefficients in (56). A closer examination shows that the condition $h \in \bigcup_{\vartheta\in\mathbb{R}} \mathcal{NRV}(\vartheta)\cap C^1$ can be relaxed to the condition $h \in C^1$ with $h'(t)/h(t) \to 0$ as $t \to \infty$. Thus, for example a C^1 regularly bounded h is allowed since it satisfies $-\infty < \liminf_{t\to\infty} th'(t)/h(t) \le \limsup_{t\to\infty} th'(t)/h(t) < \infty$.

(iii) If $\lambda = \lambda_1$ or $\lambda = \lambda_2$ is nonzero in (58), then $\lim_{t\to\infty} ty'(t)/y(t) = \pm\infty$, where the sign $+$ or $-$ is according to whether $\operatorname{sgn}\lambda = 1$ or $\operatorname{sgn}\lambda = -1$, respectively. It is known that this condition implies rapid variation. Rapidly varying functions are defined at the end of Section 2.

(iv) As far as we know, asymptotics for Poincaré–Perron solutions presented in Corollary 6 is new even in the linear case.

Example 2. Let $a(t) = A$ and $b(t) = B + f(t)$ where $|f| \in \mathcal{RV}(\vartheta)\cap C$ with $\vartheta \le 0$; if $\vartheta = 0$, we assume, in addition, $\lim_{t\to\infty} f(t) = 0$. Then, $H_1(t) = H_2(t) = f(t)$ and $|H_i\circ\ln| = |f\circ\ln| \in \mathcal{RV}(\vartheta\cdot 0) = \mathcal{SV}$. Hence, $|H_i| \in \mathcal{SV}_{\exp}$. Let $B \neq 0$ and take $\vartheta \neq -1$. Then, the Karamata theorem yields $\int_a^t (1+o(1))f(s)\,\mathrm{d}s \sim tf(t)/(\vartheta+1)$ as $t \to \infty$ when $\vartheta > -1$ and

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$\int_t^\infty (1+o(1))f(s)\,\mathrm{d}s \sim tf(t)/(-\vartheta-1)$ as $t\to\infty$ when $\vartheta<-1$. In both cases, we get, for $y\in S\cap\mathcal{NRV}_{\exp}(\eta)$, η being a root of $(\alpha-1)|\eta|^\beta + A\Phi(\eta)+B=0$,

$$y(t)=M\exp\left\{\eta t+\frac{tf(t)(1+o(1))}{(\alpha-1)(\Phi(\eta)-B/\eta)(\vartheta+1)}\right\}$$

as $t\to\infty$.

9 | PROOFS

We start with the auxiliary statements where selected properties of regularly varying functions are gathered.

Proposition 1 [1, 9, 29].

(i) If $f\in\mathcal{RV}(\vartheta)$, then $\ln f(t)/\ln t\to\vartheta$ as $t\to\infty$. It then clearly implies that $\lim_{t\to\infty}f(t)=0$ provided $\vartheta<0$, and $\lim_{t\to\infty}f(t)=\infty$ provided $\vartheta>0$.
(ii) If $f\in\mathcal{RV}(\vartheta)$, then $f^\alpha\in\mathcal{RV}(\alpha\vartheta)$ for every $\alpha\in\mathbb{R}$.
(iii) If $f_i\in\mathcal{RV}(\vartheta_i)$, $i=1,2$, $f_2(t)\to\infty$ as $t\to\infty$, then $f_1\circ f_2\in\mathcal{RV}(\vartheta_1\vartheta_2)$.
(iv) If $f_i\in\mathcal{RV}(\vartheta_i)$, $i=1,2$, then $f_1+f_2\in\mathcal{RV}(\max\{\vartheta_1,\vartheta_2\})$.
(v) If $f_i\in\mathcal{RV}(\vartheta_i)$, $i=1,2$, then $f_1f_2\in\mathcal{RV}(\vartheta_1+\vartheta_2)$.
(vi) If $f_1,\dots,f_n\in\mathcal{RV}$, $n\in\mathbb{N}$, and $R(x_1,\dots,x_n)$ is a rational function with nonnegative coefficients, then $R(f_1,\dots,f_n)\in\mathcal{RV}$.
(vii) If $L\in\mathcal{SV}$ and $\vartheta>0$, then $t^\vartheta L(t)\to\infty$, $t^{-\vartheta}L(t)\to 0$ as $t\to\infty$.
(viii) If $f\in\mathcal{RV}(\vartheta)$ and a measurable g is such that $g(t)\sim f(t)$ as $t\to\infty$, then $g\in\mathcal{RV}(\vartheta)$.
(ix) If $f\in\mathcal{RV}(\vartheta)$, $\vartheta\neq 0$, then there exists $g\in C^1$ with $g(t)\sim f(t)$ as $t\to\infty$ and such that $tg'(t)/g(t)\to\vartheta$, whence $g\in\mathcal{NRV}(\vartheta)$. Moreover, g can be taken such that $|g'|\in\mathcal{NRV}(\vartheta-1)$.
(x) Let f be eventually positive and differentiable, and let $\lim_{t\to\infty}tf'(t)/f(t)=\vartheta$, then $f\in\mathcal{NRV}(\vartheta)$.
(xi) If $|f'|\in\mathcal{RV}(\vartheta)$, $\vartheta\neq-1$, with f' being eventually of one sign, then $f\in\mathcal{NRV}(\vartheta+1)$.

The following statement (the so-called Karamata integration theorem) is of great importance in our theory.

Theorem 7 [1]. *Let $L\in\mathcal{SV}$.*

(i) If $\vartheta<-1$, then $\int_t^\infty s^\vartheta L(s)\,\mathrm{d}s\sim t^{\vartheta+1}L(t)/(-\vartheta-1)$ as $t\to\infty$.
(ii) If $\vartheta>-1$, then $\int_a^t s^\vartheta L(s)\,\mathrm{d}s\sim t^{\vartheta+1}L(t)/(\vartheta+1)$ as $t\to\infty$.
(iii) If $\int_a^\infty L(s)/s\,\mathrm{d}s$ converges, then $\widetilde{L}(t)=\int_t^\infty L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; if $\int_a^\infty L(s)/s\,\mathrm{d}s$ diverges, then $\widetilde{L}(t)=\int_a^t L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; in both cases, $L(t)/\widetilde{L}(t)\to 0$ as $t\to\infty$.

Next we present some properties of the de Hann class Π

Proposition 2 [9, 29].

(i) Auxiliary function of a function in Π is unique up to asymptotic equivalence.
(ii) If $f\in\Pi(v)$, then

$$v(t)\sim f(t)-\frac{1}{t}\int_a^t f(s)\,\mathrm{d}s \tag{63}$$

as $t\to\infty$.
(iii) If $f\in\Pi$, then $\lim_{t\to\infty}f(t)=:f(\infty)\le\infty$ exists. If the limit is infinite, then $f\in\mathcal{SV}$. If the limit is finite, then $f(\infty)-f(t)\in\mathcal{SV}$.
(iv) If $f'\in\mathcal{RV}(-1)$, then $f\in\Pi(tf'(t))$.

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The following properties of generalized $\mathcal{RV}$ functions are mostly immediate consequences of the properties of $\mathcal{RV}$ functions, see [11].

Proposition 3.

(i) $f \in \mathcal{RV}_H(\vartheta)$ *if and only if* $f(t) = H^{\vartheta}(t)L^{[H]}(t)$, *where* $L^{[H]} \in \mathcal{SV}_H$.
(ii) *If* $L^{[H]} \in \mathcal{SV}_H$ *and* $\vartheta > 0$, *then* $H^{\vartheta}(t)L^{[H]}(t) \to \infty, H^{-\vartheta}(t)L^{[H]}(t) \to 0$ *as* $t \to \infty$.
(iii) (Alternative representation theorem) $f \in \mathcal{RV}_H(\vartheta)$ *if and only if*

$$f(t) = c(t) \exp\left\{ \int_a^t \frac{H'(s)}{H(s)} \delta(s)\, \mathrm{d}s \right\},$$

$t \geq a$, *for some* $a > 0$, *where* c, δ *are measurable and* $c(t) \to c \in (0, \infty)$, $\delta(t) \to \vartheta$ *as* $t \to \infty$.
(iv) *If* $H(t) \sim KG(t)$ *as* $t \to \infty$ *for some constant* $K > 0$, *then* $\mathcal{RV}_H(\vartheta) = \mathcal{RV}_G(\vartheta)$ *for any* $\vartheta \in \mathbb{R}$.
(v) $\mathcal{RV}_{H^\gamma}(\vartheta) = \mathcal{RV}_H(\vartheta\gamma)$ *for any* $\vartheta \in \mathbb{R}$ *and* $\gamma \in (0, \infty)$.
(vi) $\mathcal{RV}_{\mathrm{id}}(\vartheta) = \mathcal{RV}(\vartheta)$ *and* $\mathcal{NRV}_{\mathrm{id}}(\vartheta) = \mathcal{NRV}(\vartheta)$.

Denote, for $\gamma \in \mathbb{R}$,

$$M_\gamma := \left| \frac{\alpha - 1 - \gamma}{\alpha - 1} \right|^{\alpha - 1} \text{ and } N_\gamma := \left| \frac{\gamma + 1 - \alpha}{\alpha} \right|^{\alpha - 1}.$$

Further, let $\eta_1^{\pm} \leq \eta_2^{\pm}$ denote the (real) roots of (30) when $\operatorname{sgn}(\alpha - 1 - \gamma) = \pm 1$ and let $\eta_1 \leq \eta_2$ denote the (real) roots of (30) when $\gamma = \alpha - 1$.

Lemma 2.

(i) *Let* $\gamma \neq \alpha - 1$. *If* $C_\gamma < 0$, *then* $\eta_1^{\pm}\eta_2^{\pm} < 0$, $|\eta_1^+| = \eta_2^-$, *and* $|\eta_1^-| = \eta_2^+ > M_\gamma$. *If* $C_\gamma = 0$, *then* $\eta_1^+ = \eta_2^- = 0$ *and* $-\eta_1^- = \eta_2^+ = M_\gamma$. *If* $C_\gamma \in (0, K_\gamma)$, *then* $\eta_1^{\pm}\eta_2^{\pm} > 0$ *and* $\eta_1^+ = |\eta_2^-| < N_\gamma < \eta_2^+ = |\eta_1^-| < M_\gamma$. *If* $C_\gamma = K_\gamma$, *then* $-\eta_1^- = -\eta_2^- = \eta_1^+ = \eta_2^+ = N_\gamma$.
(ii) *Let* $\gamma = \alpha - 1$. *Then,* $C_\gamma \leq 0$ *with* $\eta_1 = \eta_2 = 0$ *when* $C_\gamma = 0$ *while* $\eta_{1,2} = \pm(|C_\gamma|/(\alpha - 1))^{1/\beta}$ *when* $C_\gamma < 0$.
(iii) *Let* $\gamma = 0$ *or* $\gamma = 2\alpha - 2$. *Then, (30) reduces to*

$$\widehat{F}_\gamma(\lambda) := |\lambda|^\beta + \lambda \operatorname{sgn}(\gamma + 1 - \alpha) + C/(\alpha - 1) = 0, \tag{64}$$

$C := C_\gamma \in (-\infty, K]$, $K := K_\gamma = ((\alpha - 1)/\alpha)^\alpha$, *and the localization of the roots of (64) is the same as that described in part (i), now with* $M_\gamma = 1$ *and* $N_\gamma = ((\alpha - 1)/\alpha)^{\alpha - 1}$.

Proof. First consider case (iii) where we take $\gamma = 0$. Then, (64) takes the form $|\lambda|^\beta - \lambda + C/(\alpha - 1) = 0$ with $C \in (-\infty, K]$. Let $\lambda_1 \leq \lambda_2$ be its (real) roots. Let us inspect the intersections of the graphs of $x \mapsto x$ and $x \mapsto |x|^\beta + C/(\alpha - 1) =: f(x)$. First observe that $f'(((\alpha - 1)/\alpha)^{\alpha - 1}) = 1$. In other words, the graph of $f(x)$ has the same tangent as the graph of x at $x = ((\alpha - 1)/\alpha)^{\alpha - 1}$. Moreover, $f(((\alpha - 1)/\alpha)^{\alpha - 1}) = ((\alpha - 1)/\alpha)^{\alpha - 1}$ when $C = K$. Further, $\widehat{F}_\gamma(1) = \widehat{F}_\gamma(0) = 0$ when $C = 0$. These observations, along with the fact that f is convex and its graph is symmetric with respect to the vertical axis (in fact, it is a parabola-like curve) show that the following statements are valid. If $C = 0$, then $\lambda_1 = 0$ and $\lambda_2 = 1$. If $C < 0$, then $\lambda_1 < 0 < 1 < \lambda_2$. If $C = K$, then $\lambda_1 = \lambda_2 = ((\alpha - 1)/\alpha)^{\alpha - 1}$. If $C \in (0, K)$, then $0 < \lambda_1 < ((\alpha - 1)/\alpha)^{\alpha - 1} < \lambda_2 < 1$. If we take $\gamma = 2\alpha - 2$, then (64) takes the form $|\lambda|^\beta + \lambda + C/(\alpha - 1) = 0$ and hence the roots have the same properties as in the previous case (here the situation is understood symmetrically with respect to the vertical axis).

The proof of the case (i) uses the idea of transforming into (iii). Set

$$\eta = \lambda M_\gamma \text{ and } C_\gamma = \left| \frac{\gamma - \alpha + 1}{\alpha - 1} \right|^\alpha C. \tag{65}$$

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Then,

$$\begin{aligned} F_\gamma(\eta) &= |\eta|^\beta + \frac{\gamma+1-\alpha}{\alpha-1}\eta + \frac{C_\gamma}{\alpha-1} \\ &= \left|\frac{\gamma-\alpha+1}{\alpha-1}\right|^\alpha \left(|\eta|^\beta \left|\frac{\alpha-1}{\gamma+\alpha-1}\right|^{(\alpha-1)\beta} + \Phi\left(\frac{\alpha-1}{\gamma+1-\alpha}\right)\eta + \left|\frac{\alpha-1}{\gamma+1-\alpha}\right|^\alpha \frac{C_\gamma}{\alpha-1} \right) \\ &= \left|\frac{\gamma-\alpha+1}{\alpha-1}\right|^\alpha \widehat{F}_\gamma(\lambda). \end{aligned} \tag{66}$$

Hence, η is a root of $F_\gamma(\eta)=0$ if and only if λ is a root of $\widehat{F}_\gamma(\lambda)=0$. Consequently, the properties of the roots of (30) can easily be determined from the properties of the roots of $\widehat{F}_\gamma(\lambda)=0$ via the relation in (65).

Finally we prove part (ii). Let $\gamma=\alpha-1$. Then, (30) reduces to the equation $|\eta|^\beta + C_\gamma/(\alpha-1)=0$, which is readily solvable. □

Note that the transformation in part (ii) of the next lemma can be understood as a sort of half-linear version of the linear transformation of dependent variable $y=hu$ (where y is a solution of (1) and $0\neq h\in C^1$) in Equation (1) with $\alpha=2$ expressed in terms of the associated Riccati equation. Since the solution space of (1) is not generally additive, the substitution $y=hu$ does not transform (in contrast to the linear case) (1) into an equation of the same form. Rather (in the proof of Theorem 6) we use the asymptotic properties of the solutions under the particular setting and implement "asymptotic linearization," which—in terms of Riccati equation—can be described as "asymptotic quadratization."

Lemma 3.

(i) Let y be a solution of a nonoscillatory equation (1) and $w=fr\Phi(y'/y)$, where $f\in C^1$ and $f\neq 0$. Then, w satisfies the generalized Riccati differential equation

$$w' - \frac{f'(t)}{f(t)}w + f(t)p(t) + (\alpha-1)\frac{r^{1-\beta}(t)}{\Phi^{-1}(f(t))}|w|^\beta = 0 \tag{67}$$

on an interval $[b,\infty)$, where b is sufficiently large.

(ii) Let y and w be as in the part (i), where we set $f=1$, and let $v=h^\alpha w - Z$, where $Z=rh\Phi(h')$ with $h\in C^1$ and $h>0$. Then, w satisfies the generalized Riccati differential equation

$$w' + p(t) + (\alpha-1)r^{1-\beta}(t)|w|^\beta = 0 \tag{68}$$

and v satisfies the modified Riccati differential equation

$$v' + \widetilde{p}(t) + (\alpha-1)r^{1-\beta}(t)h^{-\beta}(t)F(v,Z(t)) = 0, \tag{69}$$

on an interval $[b,\infty)$, where b is sufficiently large, where

$$\widetilde{p} = h[(r\Phi(h'))' + p\Phi(h)],\quad F(v,Z) = |v+Z|^\beta - \beta\Phi^{-1}(Z)v - |Z|^\beta.$$

Lemma 4. *Let $\omega\in\{-\alpha,\alpha-2\}$, $r(t)=t^{\omega+\alpha}$, $\lim_{t\to\infty} t^{-\omega}p(t)=C\in(-\infty,K]$, K being defined in (13). If $C=K$, we assume, in addition, nonoscillation of (1). Then, $\mathcal{S}\subset\bigcup_{i\in\{1,2\}}\mathcal{NRV}(\Phi^{-1}(\lambda_i))$ with $\mathcal{S}\cap\mathcal{NRV}(\Phi^{-1}(\lambda_i))\neq\emptyset$, $i=1,2$, where $\lambda_1\le\lambda_2$ are the roots of (14). Moreover, if $p>0$, then $\mathcal{S}=\mathcal{IS}$ when $\omega=-\alpha$ and $\mathcal{S}=\mathcal{DS}$ when $\omega=\alpha-2$. If $p<0$, then $\emptyset\neq\mathcal{DS}\subset\mathcal{NRV}(\Phi^{-1}(\lambda_1))$ and $\emptyset\neq\mathcal{IS}\subset\mathcal{NRV}(\Phi^{-1}(\lambda_2))$.*

Proof. Assume that $C>0$. Let $\omega=-\alpha$. Then, $r(t)=1$, thus (9) holds, and so $\mathcal{S}=\mathcal{IS}$ by Lemma 1. Take $y\in\mathcal{S}$ and set $w=fr\Phi(y'/y)$, where $f(t)=t^{\alpha-1}$ and $r(t)=1$. Then, $w(t)=(ty'(t)/y(t))^{\alpha-1}>0$ satisfies (67), which takes the form

$$tw'(t)=H^+(t),\quad H^+(t):=-t^\alpha p(t)+(\alpha-1)w(t)(1-w^{\beta-1}(t)) \tag{70}$$

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for large t. We claim that $\lim_{t\to\infty} w(t) \in \{\lambda_1, \lambda_2\}$. First assume that there exists the limit $A := \lim_{t\to\infty} w(t)$. Clearly, $A \in [0,\infty) \cup \{\infty\}$. If $A = \infty$, then $H^+(t) \to -\infty$ as $t \to \infty$, which implies $tw'(t) \to -\infty$ as $t \to \infty$, thus w is eventually decreasing. This contradicts $A = \infty$. Now let $A \in [0,\infty) \setminus \{\lambda_1, \lambda_2\}$. Then,

$$\lim_{t\to\infty} tw'(t) = \lim_{t\to\infty} H^+(t) = -C + (\alpha-1)A(1-A^{\beta-1}) = -(\alpha-1)\left[A^\beta - A + \frac{C}{\alpha-1}\right] =: D.$$

Since λ_1, λ_2 are the only roots of (14), we get $D \neq 0$. Hence, $w'(t) \sim D/t$ as $t \to \infty$, which yields $w(t) \sim D \ln t$ as $t \to \infty$. This however contradicts finiteness of $\lim_{t\to\infty} w(t)$. It remains to examine the case when $\lim_{t\to\infty} w(t)$ does not exist. Then, w' necessarily changes its sign in any neighborhood of infinity. Thus, there are infinitely many of arbitrarily large zeroes of w'. Pick a sequence $\{t_n\}$ of (some of) these zeroes with $\lim_{n\to\infty} t_n = \infty$. From (70), we obtain $0 = H^+(t_n)$. Denote $A_* = \liminf_{t\to\infty} w(t)$, $A^* = \limsup_{t\to\infty} w(t)$, and take sequences $\{i_n\}, \{j_n\}$ such that $w(i_n) \to A_*$, $w(j_n) \to A^*$, $i_n \to \infty$, $j_n \to \infty$ as $n \to \infty$. Since the set of zeroes of w' includes the points of local maxima and local minima of w, the sequences $\{t_n\}$, $\{i_n\}$, $\{j_n\}$ can obviously be taken such that $\{i_n\}, \{j_n\} \subset \{t_n\}$. Assume that $A_* \in ([0,\infty) \cup \{\infty\}) \setminus \{\lambda_1, \lambda_2\}$. Then, from $0 = H^+(t_n)$, $0 = \lim_{n\to\infty} H^+(i_n) = -(\alpha-1)[A_*^\beta - A_* + C/(\alpha-1)] \neq 0$ provided $A_* < \infty$, while $0 = \lim_{n\to\infty} H^+(i_n) = -\infty$ provided $A_* = \infty$. In both these cases, we get a contradiction. Similarly we come to a contradiction when $A^* \in ([0,\infty) \cup \{\infty\}) \setminus \{\lambda_1, \lambda_2\}$ employing the sequence $\{j_n\}$. Consequently, $\lambda_1 = A_* < A^* = \lambda_2$; the strict inequality is here because we have supposed nonexistence of the limit $\lim_{t\to\infty} w(t)$. If $\lambda_1 = \lambda_2$ (which corresponds with $C = \gamma_\alpha$), then we are done. So we continue with $C < \gamma_\alpha$. Then, there exists a sequence $\{k_n\}$, $k_n \in [t_0,\infty)$, such that $k_n \to \infty$ and $w(k_n) \to (\lambda_1+\lambda_2)/2$ as $n \to \infty$ and $w'(k_n) < 0$. Then, we have

$$0 > k_n w'(k_n) = H^+(k_n) \to -(\alpha-1)\left[\left(\frac{\lambda_1+\lambda_2}{2}\right)^\beta - \frac{\lambda_1+\lambda_2}{2} + \frac{C}{\alpha-1}\right] > 0$$

as $k \to \infty$, a contradiction; positivity of the limit expression follows from the properties of the roots based on the inspection of the graphs of $x \mapsto x$ and $x \mapsto F(x)$ in the proof of Lemma 2. Hence, $A_* = A^* =: A \in \{\lambda_1, \lambda_2\}$ and so $\lim_{t\to\infty}(ty'(t)/y(t))^{\alpha-1} = A$, thus $y \in \bigcup_{i\in\{1,2\}} \mathcal{NRV}(\Phi^{-1}(\lambda_i))$. According to [13], the condition $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,ds = \bar{C} \in (0, (\alpha-1)^{\alpha-1}/\alpha^\alpha]$ guarantees the existence of solutions $y_i \in \mathcal{NRV}(\Phi^{-1}(\xi_i))$, where $0 < \xi_1 \le \xi_2$ are the roots $|\xi|^\beta - \xi + \bar{C} = 0$ (if $\bar{C} = (\alpha-1)^{\alpha-1}/\alpha^\alpha$, we assume, in addition, nonoscillation of (1)). Note that we have $\int_a^\infty p(s)\,ds < \infty$ thanks to $t^\alpha p(t) \to C$ as $t \to \infty$. By the L'Hospital rule, $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,ds = \lim_{t\to\infty} t^\alpha p(t)/(\alpha-1) = C/(\alpha-1)$. Hence, $\bar{C} = C/(\alpha-1)$ and $\lambda_i = \xi_i$, $i = 1,2$. Consequently, $\mathcal{S} \cap \mathcal{NRV}(\Phi^{-1}(\lambda_i)) \neq \emptyset$, $i = 1,2$.

Next we consider the case $\omega = \alpha - 2$. Then, $r(t) = t^{2\alpha-2}$. Since $p(t) \sim Ct^{\alpha-2}$ and $\alpha - 2 > -1$, we get $\int_a^\infty p(s)\,ds = \infty$, which implies $\mathcal{S} = \mathcal{DS}$ by Lemma 1. Let $y \in \mathcal{S}$. Set $w = fr\Phi(y'/y)$, where $f(t) = t^{1-\alpha}$ and $r(t) = t^{2\alpha-2}$. Then, $w(t) = \Phi(ty'(t)/y(t)) < 0$ satisfies (67), which takes the form

$$tw'(t) = H^-(t), \quad H^-(t) := -t^{2-\alpha}p(t) + (\alpha-1)w(t)(1-\Phi^{-1}(w(t))) \tag{71}$$

for large t. If the limit $B = \lim_{t\to\infty} w(t)$ exists, then $B \in (-\infty, 0] \cup \{-\infty\}$. Assume that $B = -\infty$. Then, $\lim_{t\to\infty} y(t)/(ty'(t)) = 0$. From (1) (with $r \in C^1$, which is now obviously fulfilled), we get $-r'(-y')^{\alpha-1} + (\alpha-1)r(-y')^{\alpha-2}y'' = -py^{\alpha-1}$. Hence, in view of the equality $r(t) = t^{2\alpha-2}$,

$$\frac{y''(t)y(t)}{(y'(t))^2} = \frac{-p(t)y^\alpha(t) + r'(t)(-y'(t))^{\alpha-1}y(t)}{(\alpha-1)r(-y'(t))^\alpha} = \frac{-p(t)y^\alpha(t)}{(\alpha-1)r(-y'(t))^\alpha} - \frac{r'(t)y(t)}{(\alpha-1)r(t)y'(t)}$$

$$= \frac{-p(t)y^\alpha(t)}{(\alpha-1)t^{2\alpha-2}(-y'(t))^\alpha} - \frac{(2\alpha-2)t^{2\alpha-3}y(t)}{(\alpha-1)t^{2\alpha-2}y'(t)} = -\frac{t^{2-\alpha}p(t)}{\alpha-1}\left(\frac{y(t)}{-ty'(t)}\right)^\alpha - \frac{(2\alpha-2)y(t)}{(\alpha-1)ty'(t)} \to 0$$

as $t \to \infty$. Consequently,

$$\left(\frac{y(t)}{y'(t)}\right)' = \frac{(y'(t))^2 - y(t)y''(t)}{(y'(t))^2} = 1 - \frac{y(t)y''(t)}{(y'(t))^2} \to 1$$

as $t \to \infty$, which yields $y'(t) > 0$ eventually, thus $w(t) > 0$ for large t, a contradiction with $B = -\infty$. If $B > -\infty$, then $\lim_{t\to\infty} H^-(t) = -C + (\alpha-1)B(1-\Phi^{-1}(B))$ and similarly as in the case $\omega = -\alpha$, we get a contradiction provided $B \in (-\infty, 0] \setminus \{\lambda_1, \lambda_2\}$ (in such a setting, $\lim_{t\to\infty} H^-(t) \neq 0$). The case when $\lim_{t\to\infty} w(t)$ does not exist can be proved also by

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reasoning similar to the setting $r(t)=1$. According to [13], the condition $\lim_{t\to\infty}(1/R_C(t))\int_t^\infty R_C^\alpha(s)p(s)\,\mathrm{d}s=(\alpha-1)\widehat{C}$ (where if $\widehat{C}=(\alpha-1)^{\alpha-1}/\alpha^\alpha$ we assume, in addition, nonoscillation of (1)) guarantees the existence of solutions $y_i\in\mathcal{NRV}(1/R_C)(\Phi^{-1}(\mu_i))$, with $\mu_1\le\mu_2$ being the roots $|\mu|^\beta+\mu+\widehat{C}=0$. Since $r(t)=t^{2\alpha-2}$, we have $r^{1-\beta}(t)=t^{-2}$, and so $R_C(t)=1/t$. The L'Hospital rule therefore yields $\lim_{t\to\infty}(1/R_C(t))\int_t^\infty R_C^\alpha(s)p(s)\,\mathrm{d}s=\lim_{t\to\infty}t^{2-\alpha}p(t)=C$. We have $C=(\alpha-1)\widehat{C}$, thus $\lambda_i=\mu_i$, $i=1,2$. Consequently, $\mathcal{S}\cap\mathcal{NRV}(\Phi^{-1}(\lambda_i))\ne\emptyset$, $i=1,2$.

The proof of the case $p(t)<0$ can be found in [29]. The case where $C=0$ and there is no sign condition on p is proved in [31]. □

In the next lemma, we do not consider the case $\eta>\omega$ since it would lead to $\lim_{t\to\infty}t^\alpha|p(t)|/r(t)=\lim_{t\to\infty}t^{-\omega}|p(t)|=\infty$, and by Lemma 9, it implies $\mathcal{S}\cap\bigcup_{\vartheta\in\mathbb{R}}\mathcal{RV}(\vartheta)=\emptyset$. Further, we exclude the combination $\eta<-1$, $\omega>-1$, since then $J_p<\infty, J_r<\infty$. This setting is exceptional (and somehow simpler) and requires a different approach, which is discussed later.

Lemma 5. *Let $|p|\in\mathcal{RV}(\eta)$ and $r(t)=t^{\omega+\alpha}$, where $\omega\in\{-\alpha,\alpha-2\}$ and either $\eta\le\omega<-1$ or $-1<\eta\le\omega$. If $\eta=\omega$, let, in addition, $\lim_{t\to\infty}L_p(t)=0$. Then, $\mathcal{S}\cap\mathcal{SV}=\mathcal{S}\cap\mathcal{NSV}\ne\emptyset$. If $y\in\mathcal{S}\cap\mathcal{SV}$, then*

$$(\ln y(t))'\sim\Phi^{-1}\left(\frac{-1}{\eta+1}\right)\Phi^{-1}(L_p(t))t^{(\beta-1)(\eta-\omega)-1} \tag{72}$$

as $t\to\infty$. Moreover, $y\operatorname{sgn}y'\in\Pi(g_y)$, where $g_y(t)=t|y'(t)|$, provided $\eta=\omega$.

Proof. First note that $\lim_{t\to\infty}t^{-\omega}p(t)=0$. Indeed $t^{-\omega}p(t)=t^{\eta-\omega}L_p(t)$, which tends to zero by Proposition 1-(vii) when $\eta<\omega$ resp. by the assumption of the lemma when $\eta=\omega$. From Lemma 4, we therefore get $\mathcal{S}\cap\mathcal{SV}=\mathcal{S}\cap\mathcal{NSV}\ne\emptyset$.

Let $p>0$ and $\omega=-\alpha$. In view of $r=1$, condition (9) holds, and so $\mathcal{S}=\mathcal{IS}=\mathcal{IS}_{\infty 0}\cup\mathcal{IS}_{\infty B}\cup\mathcal{IS}_{B0}$ by Lemma 1. Take $y\in\mathcal{S}\cap\mathcal{SV}$. Then, $y'>0$ and y is intermediate or subdominant (i.e., $y'(t)\to 0$ as $t\to\infty$). Indeed, from (1), we see that y' decreases and if $y'(t)\to B>0$, then $y(t)\sim Bt$ as $t\to\infty$, which contradicts $y\in\mathcal{SV}$. Integrating (1) from $t\ge b$ to ∞ ($b\ge a$ being so large that $y(t)>0, y'(t)>0, t\ge b$), we get

$$\Phi(y'(t))=\int_t^\infty p(s)\Phi(y(s))\,\mathrm{d}s\sim\frac{-1}{\eta+1}tp(t)\Phi(y(t)) \tag{73}$$

as $t\to\infty$, where the asymptotic relation follows from Theorem 7 since $|L_p|\Phi(y)\in\mathcal{SV}$. From (73), as $t\to\infty$,

$$y'(t)\sim\Phi^{-1}\left(\frac{-1}{\eta+1}\right)t^{\beta-1}\Phi^{-1}(p(t))y(t)\in\mathcal{RV}((\beta-1)(\eta+1)) \tag{74}$$

by Proposition 1-(ii),(v). For the index, we have $(\beta-1)(\eta+1)=(\beta-1)(\eta+1-\omega-\alpha)=(\beta-1)(\eta-\omega)-1$. Thus, if $\eta=\omega$, we get $y'\in\mathcal{RV}(-1)$, and so $y\operatorname{sgn}y'=y\in\Pi(g_y)$ by Proposition 2-(iv). Further, from (74) we have

$$\frac{y'(t)}{y(t)}\sim\Phi^{-1}\left(\frac{-1}{\eta+1}\right)\Phi^{-1}(L_p(t))t^{(\beta-1)(\eta-\omega)-1} \tag{75}$$

as $t\to\infty$, which yields (72).

Let $p>0$ and $\omega=\alpha-2$. Since $\alpha-2>-1$, the condition $-1<\eta\le\omega$ comes into the play, and in view of $p\in\mathcal{RV}(\eta)$, we have $\int_a^\infty p(s)\,\mathrm{d}s=\infty$, and so (8) holds by Lemma 1.

Take $y\in\mathcal{S}\cap\mathcal{SV}$. Then $t^{2\alpha-2}\Phi(y'(t))=y^{[1]}(t)\to-\infty$ as $t\to\infty$. Indeed, $y^{[1]}$ is decreasing, and if $y^{[1]}(t)\to D\in(-\infty,0)$, then $y(t)\to 0$ by (8). Therefore, $y'(t)\sim\Phi^{-1}(t^{2(1-\alpha)}D)=\Phi^{-1}(D)t^{-2}$, thus $y(t)\sim|\Phi^{-1}(D)|/t$ as $t\to\infty$, which contradicts $y\in\mathcal{SV}$. Integrating (1) from b to t ($b\ge a$ being so large that $y(t)>0, y'(t)<0, t\ge b$), we get

$$y^{[1]}(t)=y^{[1]}(b)-\int_b^t p(s)\Phi(y(s))\,\mathrm{d}s. \tag{76}$$

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Since $p\Phi(y) \in \mathcal{RV}(\eta)$, Theorem 7 yields

$$y^{[1]}(t) \sim -\int_b^t p(s)\Phi(y(s))\,\mathrm{d}s \sim \frac{-1}{\eta+1} t p(t)\Phi(y(t))$$

as $t \to \infty$, which can be rewritten as

$$|y'(t)| \sim \left|\Phi^{-1}\left(\frac{-1}{\eta+1}\right) t^{(2-2\alpha+1)(\beta-1)}\Phi^{-1}(p(t))y(t)\right| \in \mathcal{RV}((\beta-1)(2-2\alpha+1+\eta)), \tag{77}$$

where we used Proposition 1-(ii),(v). For the index of regular variation, we have $(\beta-1)(\eta+1+2-2\alpha) = (\beta-1)(\eta+1-\omega-\alpha) = (\beta-1)(\eta-\omega)-1$. Thus again, if $\eta = \omega$, then $|y'| \in \mathcal{RV}(-1)$, and so $y \operatorname{sgn} y' \in \Pi(g_y)$ by Proposition 2-(iv). Further, from (77), we get (75).

Let $p < 0$ and $\omega = -\alpha$. Take $y \in \mathcal{S} \cap \mathcal{SV}$. We know that $\mathcal{S} = \mathcal{IS} \cup \mathcal{DS}$. But slowly varying solutions cannot increase. Indeed, if $u \in \mathcal{SV} \cap \mathcal{IS}$, then $u' > 0$ is increasing, thus $u'(t) \geq B$ for some $B > 0$ and for $t \in [b, \infty)$, b being sufficiently large. Consequently, $u(t) \geq u(b) + B(t-b)$, that is, u is greater than or equal to an $\mathcal{RV}(1)$ function, a contradiction with $u \in \mathcal{SV}$, in view of Proposition 1-(vii). Hence, $y \in \mathcal{DS}$ and we have $\lim_{t\to\infty} y'(t) = 0$. Indeed, y' is negative increasing and if $y'(t) \sim D$ as $t \to \infty$ for some $D \in (-\infty, 0)$, then $y(t) \sim Dt$, a contradiction with positivity of y. The rest of the arguments for the setting $p < 0$ and $\omega = -\alpha$ are the same as in the case $p > 0$ and $\omega = -\alpha$.

Let $p < 0$ and $\omega = \alpha - 2$. Take $y \in \mathcal{S} \cap \mathcal{SV}$. Then, $y \in \mathcal{IS}$. Indeed, if $y \in \mathcal{DS} \cap \mathcal{SV}$, then $y^{[1]}$ is negative increasing and there exists the limit $\lim_{t\to\infty} y^{[1]}(t) = D \in (-\infty, 0]$. From Equation (1), we get (76), which yields $D - y^{[1]}(b) = -\int_t^\infty p(s)\Phi(y(s))\,\mathrm{d}s$, thus the integral is convergent. This contradicts with $|p|\Phi(y) \in \mathcal{RV}(\eta)$ since $\eta > -1$ and according to Theorem 7, the integral is divergent. The rest of the proof is the same as in the case $\omega = \alpha - 2$ and $p > 0$. □

Lemma 6. *Let $|p| \in \mathcal{RV}(\eta)$ and $r(t) = t^{\omega+\alpha}$, where $\omega \in \{-\alpha, \alpha-2\}$ and either $\eta \leq \omega < -1$ or $-1 < \eta \leq \omega$. If $\eta = \omega$, assume, in addition, $\lim_{t\to\infty} L_p(t) = 0$. Then, $\mathcal{S} \cap \mathcal{NSV} = \mathcal{S} \cap \mathcal{SV} \neq \emptyset$. Let $y \in \mathcal{S} \cap \mathcal{SV}$.*

(i) If $\int_a^\infty |L_p(s)|^{\beta-1} s^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}s = \infty$ (note that this can happen only when $\eta = \omega$), then

$$y(t) = \exp\left\{-\int_a^t (1+o(1))\Phi^{-1}\left(\frac{L_p(s)}{\eta+1}\right)\frac{1}{s}\,\mathrm{d}s\right\} \tag{78}$$

as $t \to \infty$. If $\omega = -\alpha$, then (18) holds. If $\omega = \alpha - 2$, then (19) holds.

(ii) If $\int_a^\infty |L_p(s)|^{\beta-1} s^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}s < \infty$ (note that the integral is automatically convergent when $\eta < \omega$), then

$$y(t) = M_y \exp\left\{\int_t^\infty (1+o(1))\Phi^{-1}\left(\frac{L_p(s)}{\eta+1}\right) s^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}s\right\} \tag{79}$$

as $t \to \infty$, where $\lim_{t\to\infty} y(t) = M_y \in (0, \infty)$. If $\omega = -\alpha$, then (18) with $\lim_{t\to\infty} y(t) = M_y$ holds. If $\omega = \alpha - 2$, then (19) with $\lim_{t\to\infty} y(t) = M_y$ holds. Moreover, $|y - M_y| \in \mathcal{SV}$ and $|L_p(t)|^{\beta-1} = o(|y(t) - M_y|)$ as $t \to \infty$ when $\eta = \omega$, while

$$|y(t) - M_y| \sim \frac{M_y}{|\eta+1|^{\beta-1}(\beta-1)(\omega-\eta)} |L_p(t)|^{\beta-1} t^{(\beta-1)(\eta-\omega)} \tag{80}$$

as $t \to \infty$ when $\eta < \omega$.

Proof. The nonemptiness of $\mathcal{S} \cap \mathcal{NSV} \neq \emptyset$ follows from Lemma 5. Let $p > 0$ and $\omega = -\alpha$. Take $y \in \mathcal{S} \cap \mathcal{SV}$. Then, $y \in \mathcal{IS}$ and $y^{[1]}(t) \to 0$ for $t \to \infty$ as shown in the proof of Lemma 5. Let $\int_a^\infty L_p^{\beta-1}(s) s^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}s = \infty$. Then, necessarily $\eta = \omega$, otherwise the integral would be convergent by Theorem 7. Integrating (72) from b to t ($b \geq a$ being so large that $y(t) > 0, y'(t) > 0, t \geq b$), we get

$$\ln y(t) = \ln y(b) + \int_b^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\eta+1}\right)\Phi^{-1}(L_p(s))\frac{1}{s}\,\mathrm{d}s \tag{81}$$

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as $t \to \infty$. We will need the following auxiliary observation. If there is a positive $f \in C$ such that $\int_a^\infty f(s)\,\mathrm{d}s = \infty$, then for a given $A \in \mathbb{R}$ and $\varepsilon_1(t) \to 0$, there exist $\varepsilon_2(t) \to 0$ and $\varepsilon_3(t) \to 0$ such that

$$A + \int_b^t (1+\varepsilon_1(s))f(s)\,\mathrm{d}s = (1+\varepsilon_3(t))\int_a^t (1+\varepsilon_1(s))f(s)\,\mathrm{d}s = \int_a^t (1+\varepsilon_2(s))f(s)\,\mathrm{d}s$$

as $t \to \infty$. Hence, relation (81) can be written as

$$\ln y(t) = \int_a^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\eta+1}\right)\Phi^{-1}(L_p(s))\frac{1}{s}\,\mathrm{d}s$$

as $t \to \infty$. Taking exp of both sides, we get (78). Clearly, $y(t) \to \infty$ as $t \to \infty$. If $\int_a^\infty L_p^{\beta-1}(s)s^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}s < \infty$, then we integrate (72) from t to ∞ to obtain

$$-\ln\frac{y(t)}{M_y} = \int_t^\infty (1+o(1)))\Phi^{-1}\left(\frac{-1}{\eta+1}\right)\Phi^{-1}(L_p(s))s^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}s$$

as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t)$ is obviously a finite number. Formula (79) easily follows. If $\eta = \omega$, then $|y - M_y| \in \mathcal{SV}$ by Proposition 2-(iii) since $y \operatorname{sgn} y' \in \Pi(g_y)$. Moreover, integration of (73) yields

$$|y(t) - M_y| \sim \frac{1}{|\eta+1|^{\beta-1}}\int_t^\infty |L_p(s)|^{\beta-1}\frac{1}{s}y(s)\,\mathrm{d}s \sim \frac{M_y}{|\eta+1|^{\beta-1}}\int_t^\infty |L_p(s)|^{\beta-1}\frac{1}{s}\,\mathrm{d}s \tag{82}$$

as $t \to \infty$. From Theorem 7, we have $L_p^{\beta-1}(s) = o\left(\int_t^\infty |L_p(s)|^{\beta-1}/s\,\mathrm{d}s\right)$ as $t \to \infty$, which, in view of (82), implies $L_p^{\beta-1}(t) = o(|y(t) - M_y|)$ as $t \to \infty$. If $\eta < \omega$, then (73) and Theorem 7 yield

$$|y(t) - M_y| \sim \frac{1}{|\eta+1|^{\beta-1}}\int_t^\infty |L_p(s)|^{\beta-1}s^{(\beta-1)(\eta-\omega)-1}y(s)\,\mathrm{d}s \sim \frac{M_y}{|\eta+1|^{\beta-1}(\beta-1)(\omega-\eta)}L_p^{\beta-1}(t)t^{(\beta-1)(\eta-\omega)}$$

as $t \to \infty$. Similarly, we proceed in the other cases (which are formed by distinguishing the sign of p and the choice of $\omega \in \{-\alpha, \alpha-2\}$). □

Lemma 7. *Let $|p| \in \mathcal{RV}(\eta)$, where $\eta < -1$, and $r(t) = t^{2\alpha-2}$. Then, $\mathcal{S} \cap \mathcal{NSV} \neq \emptyset$. Let $y \in \mathcal{S} \cap \mathcal{NSV}$. Then, (23) holds and all the subclasses are nonempty. If $y \in \mathcal{IS}_{B0}$ or $y \in \mathcal{DS}_{B0}$, then it obeys formula (79). If $y \in \mathcal{IS}_{BB}$ or $y \in \mathcal{DS}_{BB}$, then*

$$|M_y - y(t)| \sim |N_y|^{\beta-1}\frac{1}{t} \quad \text{and} \quad \left|N_y - y^{[1]}(t)\right| \sim \frac{M_y^{\alpha-1}}{|\eta+1|}t|p(t)| \tag{83}$$

as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t) \in (0,\infty)$ and $N_y = \lim_{t\to\infty} y^{[1]}(t) \in \mathbb{R}\setminus\{0\}$.

Proof. Since $t^{2-\alpha}p(t) \in \mathcal{RV}(2-\alpha+\eta)$ and for the index we have $2-\alpha+\eta < 1-\alpha < 0$, we get $\lim_{t\to\infty} t^{2-\alpha}p(t) = 0$. Thus, Lemma 4 yields $\mathcal{S} \cap \mathcal{NSV} \neq \emptyset$. The rest of the statement follows from [32, Theorem 3]. □

Lemma 8. *Let $L(t) := t^{-\omega}p(t) - C \to 0$, $C \in (-\infty, K)$, $\omega \in \{-\alpha, \alpha-2\}$, $|L| \in \mathcal{RV}(\tau)$, $\tau \le 0$, and $r(t) = t^{\omega+\alpha}$. Assume that $\lambda_1 < \lambda_2$ are the roots of (14) and $\vartheta_i = \Phi^{-1}(\lambda_i)$, $i = 1,2$; if $C = 0$, we consider only the nonzero root. If $y_i \in \mathcal{S} \cap \mathcal{NRV}(\vartheta_i)$, $i = 1,2$, then*

$$L_{y_i}\operatorname{sgn}(L_{y_i})' \in \Pi(\mathrm{g}_i),\ i = 1,2,\ \text{where } \mathrm{g}_i(t) = t|L'_{y_i}(t)|, \tag{84}$$

provided $\tau = 0$. For any $y_i \in \mathcal{S} \cap \mathcal{NRV}(\vartheta_i)$ the following holds:

(i) If $\int_a^\infty |L(s)|/s\,\mathrm{d}s = \infty$ (note that this can happen only when $\tau = 0$), then

$$y_i(t) = t^{\vartheta_i}\exp\left\{-\int_a^t \frac{(1+o(1))L(s)}{(\alpha-1)(\Phi(\vartheta_i) - C_\gamma/\vartheta_i)s}\,\mathrm{d}s\right\} \tag{85}$$

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as $t \to \infty$. *If* $L > 0$, *then,* $L_{y_1}(t) \nearrow \infty$ *and* $L_{y_2}(t) \searrow 0$. *If* $L < 0$, *then,* $L_{y_1}(t) \searrow 0$ *and* $L_{y_2}(t) \nearrow \infty$. *If* $C \neq 0$, *then* $y_i^{[1]} \searrow 0$, $i = 1, 2$, *when* $p > 0$ *and* $\omega = -\alpha$, *while* $y_i^{[1]} \searrow -\infty$, $i = 1, 2$, *when* $p > 0$ *and* $\omega = \alpha - 2$, *and* $y_1^{[1]}(t) \nearrow 0$, $y_2^{[1]}(t) \nearrow \infty$ *when* $p < 0$. *If* $C = 0$ *and* $\omega = -\alpha$, *then* $y_2^{[1]}(t) \searrow 0$ *when* $p > 0$, *while* $y_2^{[1]}(t) \nearrow \infty$ *when* $p < 0$. *If* $C = 0$ *and* $\omega = \alpha - 2$, *then* $y_1^{[1]}(t) \searrow -\infty$ *when* $p > 0$, *while* $y_1^{[1]}(t) \nearrow 0$ *when* $p < 0$.

(ii) *If* $\int_a^\infty |L(s)|/s \, \mathrm{d}s < \infty$, *then*

$$y_i(t) = t^{\vartheta_i} M_i \exp\left\{ \int_t^\infty \frac{(1 + o(1))L(s)}{(\alpha - 1)(\Phi(\vartheta_i) - C_\gamma/\vartheta_i + \tau|\vartheta_i|^{\alpha-2})s} \, \mathrm{d}s \right\} \tag{86}$$

as $t \to \infty$, *where* $M_i = \lim_{t\to\infty} L_{y_i}(t) \in (0, \infty)$, $i = 1, 2$. *As for monotonicity of* $L_{y_i}(t)$, *the same observations as in (i) hold with the modification that the limit values are replaced by* M_i. *As for the asymptotic behavior of* $y_i^{[1]}$, *it is the same as in (i) when* $C \neq 0$. *If* $C = 0$, *then monotonicity of* $y_1^{[1]}$ *and* $y_2^{[1]}$ *is also the same with the modification that the limit values are replaced by nonzero real numbers. Moreover, if* $\tau = 0$, *then* $|M_i - L_{y_i}| \in \mathcal{SV}$ *and*

$$|L(t)| = o(|M_i - L_{y_i}(t)|) \tag{87}$$

as $t \to \infty$, $i = 1, 2$, *while if* $\tau < 0$, *then*

$$|M_i - L_{y_i}(t)| \sim \frac{M_i |L(t)|}{|\tau|(\alpha - 1)|\Phi(\vartheta_i) - C_\gamma/\vartheta_i + \tau|\vartheta_i|^{\alpha-2}|} \tag{88}$$

as $t \to \infty$, $i = 1, 2$.

Proof. The part concerning the asymptotic formulas is in fact corollary of Theorem 6. Indeed, if $r(t) = t^{\omega+\alpha}$, then $G_1 = G_2 = L$, $\tau_1 = \tau_2 = \tau$, and $\xi_i = \vartheta_i$. Since relation (84) follows from the observations around (112), which are contained in the proof of Theorem 6, we refer the reader to that part.

It remains to analyze the behavior of $y_i^{[1]}$. First note that because of the sign assumptions on p, for $y \in \mathcal{S}$, the expression $y^{[1]}$ is always eventually monotone as it follows from Equation (1).

Let $C \neq 0$. Then, $0 \neq |\vartheta_i| \neq 1$ by Lemma 2. Take $y \in \mathcal{NRV}(\vartheta) \cap \mathcal{S}$, where $\vartheta = \vartheta_1$ or $\vartheta = \vartheta_2$. Then, because of $\vartheta \neq 0$, we get that $\lim_{t\to\infty} y(t) = 0$ if $y \in \mathcal{DS}$ (ϑ is negative here) or $\lim_{t\to\infty} y(t) = \infty$ if $y \in \mathcal{IS}$ (ϑ is positive here). Assume by a contradiction that $y^{[1]}(t) \sim N_y \in \mathbb{R} \setminus \{0\}$ as $t \to \infty$. Then, $y'(t) \sim \Phi^{-1}(N_y) t^{-(\omega+\alpha)(\beta-1)}$ as $t \to \infty$. If $\omega = -\alpha$, then, as $t \to \infty$, $y(t) \sim \Phi^{-1}(N_y)t \in \mathcal{RV}(1)$, a contradiction with $y \in \mathcal{RV}(\vartheta)$, $\vartheta \neq 1$. If $\omega = \alpha - 2$, then $-y(t) + \lim_{s\to\infty} y(s) \sim \Phi^{-1}(N_y)/t$ as $t \to \infty$, where $\lim_{s\to\infty} y(s)$ must be zero (indeed, if $\lim_{s\to\infty} y(s) \in (0, \infty)$, then $y \in \mathcal{SV}$, a contradiction with $\vartheta \neq 0$). Thus, $y(t) \sim -\Phi^{-1}(N_y)/t \in \mathcal{RV}(-1)$, a contradiction with $|\vartheta| \neq 1$. Consequently, $\lim_{t\to\infty} y^{[1]}(t) = 0$ (provided $y^{[1]}$ is positive decreasing or negative increasing) or $\lim_{t\to\infty} |y^{[1]}| = \infty$ (provided it is positive increasing or negative decreasing).

Now consider the case $C = 0$. Then, one root of (14) is always zero whereas the nonzero root is $\lambda_2 = 1$ when $\omega = -\alpha$ and $\lambda_1 = -1$ when $\omega = \alpha - 2$. The expression L reduces to $L(t) = t^{-\omega} p(t)$. Since $|L| \in \mathcal{RV}(\tau)$, we have $|p| \in \mathcal{RV}(\omega + \tau)$ by Proposition 1-(v). We will use the transformation into the reciprocal equation. Set $u = |y^{[1]}|$. Then, u satisfies the reciprocal equation

$$(\overline{r}(t)\Phi^{-1}(u'))' + \overline{p}(t)\Phi^{-1}(u) = 0, \tag{89}$$

where

$$\overline{r}(t) = |p(t)|^{1-\beta} \quad \text{and} \quad \overline{p}(t) = r^{1-\beta}(t) \operatorname{sgn} p(t) = t^{(1-\beta)(\omega+\alpha)} \operatorname{sgn} p(t);$$

actually we have $u \in \mathcal{S}^{(89)}$. Looking at the nonlinearities, we see that the roles of α and β with respect to (1) are interchanged; indeed, $\Phi^{-1}(s) = |s|^{\beta-1} \operatorname{sgn} s$. We have $|\overline{p}| \in \mathcal{RV}((\omega + \alpha)(1 - \beta))$ and $\overline{r} \in \mathcal{RV}((\omega + \tau)(1 - \beta))$. Further,

$$\frac{L_{\overline{p}}(t)}{L_{\overline{r}}(t)} = \frac{t^\beta |\overline{p}(t)|}{\overline{r}(t)} = \frac{t^\beta r^{1-\beta}(t)}{|p(t)|^{1-\beta}} = \frac{t^{\beta+(1-\beta)(\omega+\alpha)}}{(t^{\omega+\tau}|L_p(t)|)^{1-\beta}} = (t^\tau |L_p(t)|)^{\beta-1} \to 0$$

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as $t \to \infty$ and

$$\left(\frac{t|\overline{p}|}{\overline{r}(t)}\right)^{\alpha-1} = \frac{|L_p(t)|}{t^{1-\tau}}. \tag{90}$$

Thus, we may apply Theorem 5 and Corollary 1 to reciprocal equation (89). The "reciprocal analogy" of algebraic equation (14) under our setting (recall that $C = 0$) takes the form $|\eta|^\alpha - (\omega+1)\eta = 0$. Applying Theorem 5 to the solution u of (89), we find that $u \in \mathcal{NSV} \cup \mathcal{NRV}(\omega+1)$. We claim that the solution u, which through the substitution relation $u = |y^{[1]}|$ corresponds with $y \in \mathcal{S} \cap \mathcal{RV}(\lambda_0)$ with $\lambda_0 \neq 0$, is in $\mathcal{NSV}$; note that $\lambda_0 = 1$ when $\omega = -\alpha$ and $\lambda_0 = -1$ when $\omega = \alpha - 2$. Indeed, assume that $u \in \mathcal{S}^{(89)} \cap \mathcal{NRV}(\omega+1)$. Then, if $\omega = -\alpha$, applying Proposition 1-(ii),(v), we get $|y'|^{\alpha-1} = |y^{[1]}| \in \mathcal{NRV}(1-\alpha)$, and so $|y'| \in \mathcal{NRV}(-1)$. Similarly, if $\omega = \alpha - 2$, then $t^{2\alpha-2}|y'(t)|^{\alpha-1} = |y^{[1]}(t)| \in \mathcal{NRV}(\alpha-1)$, and so again $|y'| \in \mathcal{NRV}(-1)$. In both cases we get a contradiction with $|y'(t)| \sim |\lambda_0|y(t)/t \in \mathcal{RV}(\lambda_0 - 1)$ where the last relation follows from $y \in \mathcal{NRV}(\lambda_0)$. In view of (90), applying Corollary 1 and recalling that in the reciprocal equation the roles of α and β are interchanged, we get that if $\int_a^\infty |L_p(s)|/s^{1-\tau}\,\mathrm{d}s = \infty$ (which can happen only for $\tau = 0$), then $\int_a^\infty |s\overline{p}(s)/\overline{r}(s)|^{\alpha-1}\,\mathrm{d}s = \infty$, and therefore $y^{[1]}(t) = -u(t) \searrow -\infty$ as $t \to \infty$ when $p > 0$ and $y^{[1]}(t) = -u(t) \nearrow 0$ as $t \to \infty$ when $p < 0$. Further, under the assumption $\int_a^\infty |L_p(s)|/s^{1-\tau}\,\mathrm{d}s < \infty$, we get that $y^{[1]}(t) = -u(t) \searrow M \in (-\infty, 0)$ as $t \to \infty$ when $p > 0$ and $y^{[1]}(t) = -u(t) \nearrow M \in (-\infty, 0)$ as $t \to \infty$ when $p < 0$. □

Proof of Theorem 1. Let y be a solution of (1). Set $s = \varphi(t)$ and $x(s) = y(t)$, where $\varphi \in C^1$ and $\varphi' \neq 0$. Then, $\frac{\mathrm{d}}{\mathrm{d}t} = \varphi'(t)\frac{\mathrm{d}}{\mathrm{d}s}$ and Equation (1) is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\widehat{r}(s)\Phi\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)\right) + \widehat{p}(s)\Phi(x) = 0, \tag{91}$$

where

$$\widehat{r} = (r\circ\varphi^{-1})\Phi(\varphi'\circ\varphi^{-1}), \quad \widehat{p} = \frac{p\circ\varphi^{-1}}{\varphi'\circ\varphi^{-1}}. \tag{92}$$

It holds that

$$x^{[1]}(s) = \widehat{r}(s)\Phi\left(\frac{\mathrm{d}x}{\mathrm{d}s}(s)\right) = r(t)\Phi(y'(t)) = y^{[1]}(t). \tag{93}$$

Assume that (9) holds. Let us work on an interval $[c, \infty)$, $c \geq a$. We choose $c > a$ when the zero value of $R_D(a)$ could cause a problem (e.g., in denominator). Such a setting has no influence on the results. Set $\varphi = R_D$. Then, $\widehat{r} = 1$, $\widehat{p} = (pr^{\beta-1})\circ R_D^{-1}$, and the interval $[c, \infty)$ is transformed onto $[\widehat{c}, \infty)$. From condition (13), we get $s^\alpha \widehat{p}(s) \to C$ as $s \to \infty$. Thus, the assumptions of Lemma 4 are fulfilled for Equation (91). Take $y \in \mathcal{S}$. Then, x given by $x(s) = y(t)$ satisfies $x \in \mathcal{S}^{(91)}$; note that if $y \in \mathcal{IS}$ ($y \in \mathcal{DS}$), then $x \in \mathcal{IS}^{(91)}$ ($x \in \mathcal{DS}^{(91)}$) by (93). From Lemma 4, we get $\mathcal{S}^{(91)} \subset \bigcup_{i\in\{1,2\}} \mathcal{NRV}(\Phi^{-1}(\lambda_i))$ with $\mathcal{S}^{(91)} \cap \mathcal{NRV}(\Phi^{-1}(\lambda_i)) \neq \emptyset, i = 1, 2$. Moreover, $\mathcal{S} = \mathcal{IS}$ when $\widehat{p} > 0$ whereas $\emptyset \neq \mathcal{DS}^{(91)} \subset \mathcal{NRV}(\Phi^{-1}(\lambda_1)), \emptyset \neq \mathcal{IS}^{(91)} \subset \mathcal{NRV}(\Phi^{-1}(\lambda_2))$ when $\widehat{p} < 0$. In view of the fact that the solutions x and y of (91) and (1), respectively, are related by $x = y\circ R_D^{-1}$, we get the statement.

Now we assume that (10) holds. Set $\varphi = 1/R_C$. Then, $[a, \infty)$ is transformed onto an interval of the same form. Further, $R_C\circ(1/R_C)^{-1}(s) = 1/s$ and $\varphi'(t) = r^{1-\beta}(t)/R_C^2(t)$. Hence, $\widehat{r}(s) = (R_C^{2(1-\alpha)}\circ(1/R_C)^{-1})(s) = s^{2\alpha-2}$ and $\widehat{p} = (R_C^2 pr^{\beta-1})\circ(1/R_C)^{-1}$. Condition (13) in terms of the transformed equation reads as $\lim_{s\to\infty} s^{2-\alpha}\widehat{p}(s) = C$, thus the assumptions of Lemma 4 with $\omega = \alpha - 2$ are fulfilled for (91). The rest of the proof for the case (10) is analogous to that in the case when (9) holds. □

Lemma 9. *Let $|p| \in \mathcal{RV}(\delta)$ with $\delta \neq -1$ and $r \in \mathcal{RV}(\gamma)$ with $\gamma \neq \alpha - 1$. If $\mathcal{S} \cap \mathcal{RV}(\vartheta) \neq \emptyset$, where $\eta = \Phi(\vartheta)$ is a real root of (30), then (29) holds and $\delta + \alpha \leq \gamma$.*

Proof. Let $y \in \mathcal{S} \cap \mathcal{NRV}(\Phi^{-1}(\eta))$. Assume that $\gamma < \alpha - 1$ and $\delta < -1$. Then, (9) holds. Denote $\varrho_\gamma := (\alpha - 1 - \gamma)/(\alpha - 1)$. Set $\xi = \eta|\varrho_\gamma|^{1-\alpha}$ and $\overline{C} = (\alpha-1)^{\alpha-1}/|\gamma - \alpha + 1|^\alpha C_\gamma$, $C_\gamma \in (-\infty, K_\gamma]$, K_γ being defined in (29). Then, we have $|\xi|^\beta - \xi + \overline{C} = |\varrho_\gamma|^{-\alpha}(|\eta|^\beta - \varrho_\gamma\eta + C_\gamma/(\alpha-1)) = 0$. Thus, ξ is a real root of $|\xi|^\beta - \xi + \overline{C} = 0$, where $\overline{C} \in (-\infty, (\alpha-1)^{\alpha-1}/\alpha^\alpha]$; note

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that with $\overline{C} > (\alpha-1)^{\alpha-1}/\alpha^{\alpha}$, the equation $|\xi|^{\beta} - \xi + \overline{C} = 0$ would not have real roots. Moreover, since $R_D \in \mathcal{RV}(\varrho_\gamma)$, by Proposition 1-(iii),(v), we have

$$y \circ R_D^{-1} \in \mathcal{NRV}\left(\frac{\Phi^{-1}(\eta)}{\varrho_\gamma}\right) = \mathcal{NRV}(\Phi^{-1}(\xi)), \text{ i.e., } y \in \mathcal{NRV}_{R_D}(\Phi^{-1}(\xi)).$$

According to [13], this implies that

$$R_D^{\alpha-1}(t)\int_t^{\infty} p(s)\,\mathrm{d}s = \overline{C}. \tag{94}$$

In fact, this statement on necessity assumes in [13] that $\overline{C} < (\alpha-1)^{\alpha-1}/\alpha^{\alpha}$, but it is easy to see that in case of equality it works as well. Note that if $\overline{C} > (\alpha-1)^{\alpha-1}/\alpha^{\alpha}$, that is, $C_\gamma > K_\gamma$, then (1) is oscillatory, [7]. From the Karamata integration theorem, we get

$$R_D(t) \sim \frac{1}{\varrho_\gamma} t r^{1-\beta}(t) \text{ and } \int_t^{\infty} p(s)\,\mathrm{d}s \sim \frac{1}{|\delta+1|} t p(t)$$

as $\to \infty$. Hence,

$$\frac{t^{\alpha}p(t)}{r(t)} \sim |\delta+1|\varrho_\gamma^{\alpha-1} R_D^{\alpha-1}(t)\int_t^{\infty} p(s)\,\mathrm{d}s \sim |\delta+1|\varrho_\gamma^{\alpha-1}\overline{C}$$

as $t \to \infty$. For the constant on the right-hand side of the last relation, we have $|\delta+1|\varrho_\gamma^{\alpha-1}\overline{C} = C_\gamma|\delta+1|/(\alpha-1-\gamma)$. If $\overline{C} \neq 0$, then, as $t \to \infty$, $|p(t)| \sim C_\gamma|\delta+1|t^{-\alpha}r(t)/(\alpha-1-\gamma) \in \mathcal{RV}(\gamma-\alpha)$. Hence, $\delta = \gamma-\alpha$, and we get $|\delta+1|\varrho_\gamma^{\alpha-1}\overline{C} = C_\gamma$, which implies (29). If $\overline{C} = 0$, then we again get (29), now with $C_\gamma = 0$.

Similarly we can prove the case $\gamma > \alpha - 1$ and $\delta > -1$, where we utilize the following implication (see [13]): If $y \in \mathcal{S} \cap \mathcal{NRV}_{1/R_C}(\Phi^{-1}(\mu))$, where μ is a real root of $|\mu|^{\beta} + \mu + \widetilde{C} = 0$, then

$$\frac{1}{R_C(t)}\int_t^{\infty} R_C^{\alpha}(s)p(s)\,\mathrm{d}s = (\alpha-1)\widetilde{C}. \tag{95}$$

We employ also the fact that $1/R_C \in \mathcal{RV}(|\varrho_\gamma|)$. The details are left to the reader.

If $\gamma > \alpha - 1$ and $\delta < -1$, then $\alpha+\delta-\gamma < 0$, and in view of $t^{\alpha}|p(t)|/r(t) \in \mathcal{RV}(\alpha+\delta-\gamma)$, condition (29) with $C_\gamma = 0$ holds. Similarly, if $\gamma < \alpha - 1$ and $\delta > -1$, then $\lim_{t\to\infty} t^{\alpha}p(t)/r(t) = -\infty$ provided $p < 0$. But then all eventually positive solutions of (1) are in the de Haan classes Γ and Γ_- (which are subsets of rapidly varying solutions), see [33], which contradicts to $\mathcal{S} \cap \mathcal{RV}(\vartheta) \neq \emptyset$. For $p > 0$ we obtain $\int_a^{\infty} r^{1-\beta}(s)\,\mathrm{d}s = \infty = \int_a^{\infty} p(s)\,\mathrm{d}s$, which yields oscillation of (1) (see [7, chapter 1]), thus $\mathcal{S} = \emptyset$, again a contradiction. Finally, note that the inequality $\delta + \alpha \leq \gamma$ must hold, otherwise we would get $\lim_{t\to\infty} t^{\alpha}|p(t)|/r(t) = \infty$, a contradiction with the fact that we have obtained the finite limit $\lim_{t\to\infty} t^{\alpha}p(t)/r(t)$ under the assumptions of the lemma. □

Proof of Remark 3. Making the substitution $s = Q^{\sigma}(t)$ and $x(s) = y(t)$, where y is a solution of (1) we find that $\mathcal{S} \cap \mathcal{NRV}_{Q^{\sigma}}(\vartheta) \neq \emptyset$ implies $\mathcal{S}^{(91)} \cap \mathcal{NRV}(\vartheta) \neq \emptyset$. Moreover, $\widehat{p} = (pr^{\beta-1}) \circ R_D^{-1} = (Q^{\sigma\omega+\alpha}pr^{\beta-1}) \circ (Q^{\sigma})^{-1}$ when (9) holds while $\widehat{p} = (R_C^2 pr^{\beta-1}) \circ (1/R_C)^{-1} = (Q^{\sigma\omega+\alpha}pr^{\beta-1}) \circ (Q^{\sigma})^{-1}$ when (10) holds. Hence, from (15), we get that $|\widehat{p}| \in \mathcal{RV}(\eta)$ with $\eta \neq -1$. Further, since $\widehat{r} = 1$ when (9) holds while $\widehat{r}(s) = s^{2\alpha-2}$ when (10) holds, we have $\widehat{r} \in \mathcal{RV}(\omega+\alpha)$, where $\omega + \alpha \neq \alpha - 1$. Thus, all the assumptions of Lemma 9 are fulfilled for (91). Consequently, recalling that $(Q^{\alpha}pr^{\beta-1})(t) = (R_D^{\alpha}pr^{\beta-1})(t) = s^{\alpha}\widehat{p}(s)/\widehat{r}(s)$ when (9) holds while $(Q^{\alpha}pr^{\beta-1})(t) = (R_C^{-\alpha}R_C^{2\alpha-2}R_C^2 pr^{\beta-1})(t) = s^{\alpha}\widehat{p}(s)/\widehat{r}(s)$ when (10) holds, we obtain (29), which in our setting reads as

$$\lim_{t\to\infty} Q^{\alpha}(t)p(t)r^{\beta-1}(t) = \lim_{s\to\infty}\frac{s^{\alpha}\widehat{p}(s)}{\widehat{r}(s)} = C_{\omega+\alpha} = C \in (-\infty, K_{\omega+\alpha}] = (-\infty, K].$$

Thus, we get (13). Lemma 9 further yields $\eta + \alpha \leq \omega + \alpha$, which implies $\eta \leq \omega$. □

Proof of Theorem 2. Assume that (9) holds. Let us transform Equation (1) into (91), where we set $\varphi = R_D$. Condition (15) reads as $|p|r^{\beta-1} \in \mathcal{RV}_{R_D}(\eta)$. Since $\widehat{p} = (pr^{\beta-1}) \circ R_D^{-1}$, we get $|\widehat{p}| \in \mathcal{RV}(\eta)$. Moreover, $\lim_{s\to\infty} L_{\widehat{p}}(s) = \lim_{s\to\infty} s^{-\eta}\widehat{p}(s) =$

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$\lim_{t\to\infty} R_D^{-\alpha}(t)p(t)r^{\beta-1}(t) = 0$, where the last equality follows from condition (16). Take $y \in \mathcal{S}\cap\mathcal{NSV}_{R_D}$ noting that the set is nonempty by Theorem 1. Thanks to the relation $x = y\circ R_D^{-1}$, we get $x \in \mathcal{S}^{(91)}\cap\mathcal{NSV}$ and the assumptions of Lemma 6 with $\omega = -\alpha$ are fulfilled. Using the substitution $v = R_D^{-1}(u)$, we obtain

$$\int_{\widehat{c}}^{s} \Phi^{-1}(L_{\widehat{p}}(u))u^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}u = \int_{\widehat{c}}^{s} \Phi^{-1}(u\widehat{p}(u))\,\mathrm{d}u = \int_{c}^{t} \Phi^{-1}(R_D(v)p(v)r^{\beta-2}(v))\,\mathrm{d}v, \tag{96}$$

where $c = R_D^{-1}(\widehat{c})$ ($\widehat{c}$ being some positive number) and $t = R_D^{-1}(s)$. Hence, $\int_a^\infty (R_D(s)|p(s)|r^{\beta-2}(s))^{\beta-1}\,\mathrm{d}s = \infty$ implies $\int_{\widehat{a}}^\infty |L_{\widehat{p}}(u)|^{\beta-1}u^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}u = \infty$; observe that the divergence of the integral can occur only when $\eta = \omega$ since we assume $\eta \le \omega$. Applying Lemma 6, we get

$$x(s) = \exp\left\{-\int_{\widehat{a}}^{s}(1+o(1))\Phi^{-1}\left(\frac{u\widehat{p}(u)}{\eta+1}\right)\mathrm{d}u\right\}$$

with $x(s)\nearrow\infty$ and $x^{[1]}(s)\searrow 0$ as $s\to\infty$; recall that here $\eta = -\alpha$. In view of (96), $x = y\circ R_D^{-1}$, and (93), we obtain (17) with $Q = R_D$ and (18). Similarly we proceed to obtain (20) with $Q = R_D$ and general $\eta \le \omega$ under the condition $\int_a^\infty (R_D(s)|p(s)|r^{\beta-2}(s))^{\beta-1}\,\mathrm{d}s < \infty$. Let $\eta = \omega$. Then, $|y - M_y| \in \mathcal{SV}_{R_D}$ is implied by $|x - M_x| \in \mathcal{SV}$, where $M_x = M_y$; the relation $|x - M_x| \in \mathcal{SV}$ follows from Lemma 6. Formula (21) is obtained from $|L_{\widehat{p}}(s)|^{\beta-1} = o(|x(s) - M_x|)$ as $s\to\infty$ (see Lemma 6) and the identity $s^\alpha\widehat{p}(s) = R_D^\alpha(t)p(t)r^{\beta-1}(t)$. If $\eta < \omega$, then formula (22) is implied by the identity

$$|L_{\widehat{p}}(s)|s^{(\eta-\omega)} = R_D^{(\eta-\omega)(\sigma-1)+\alpha}(t)|p(t)|r^{\beta-1}(t)$$

and formula (80).

Assume that (10) holds. Let us again transform equation (1) into (91), where we now set $\varphi = 1/R_C$. Then $\widehat{r}(s) = s^{2\alpha-2}$ and $\widehat{p} = (R_C^2 pr^{\beta-1})\circ(1/R_C)^{-1}$. Condition (15) reads as $R_C^2|p|r^{\beta-1} \in \mathcal{RV}_{1/R_C}(\eta)$. Hence, $\widehat{p}\in\mathcal{RV}(\eta)$. Moreover,

$$\lim_{s\to\infty} L_{\widehat{p}}(s) = \lim_{s\to\infty} s^{-\eta}\widehat{p}(s) = \lim_{t\to\infty} R_C^\eta(t)R_C^2(t)p(t)r^{\beta-1}(t) = \lim_{t\to\infty} R_C^{\eta+2}(t)p(t)r^{\beta-1}(t) = 0,$$

where the last equality follows from condition (16). Take $y \in \mathcal{S}\cap\mathcal{NSV}_{1/R_C}$. Thanks to the relation $x = y\circ(1/R_C)^{-1}$, we get $x \in \mathcal{S}^{(91)}\cap\mathcal{NSV}$ and the assumptions of Lemma 6 with $\omega = \alpha - 2$ are fulfilled. Using the substitution $v = (1/R_C)^{-1}(u)$, we obtain

$$\int_{\widehat{c}}^{s} \Phi^{-1}(L_{\widehat{p}}(u))u^{(\beta-1)(\eta-\omega)-1}\,\mathrm{d}u = \int_{\widehat{c}}^{s} \Phi^{-1}(u^{3-2\alpha}\widehat{p}(u))\,\mathrm{d}u = \int_{c}^{t} \Phi^{-1}(R_C(v)p(v)r^{\beta-2}(v))\,\mathrm{d}v, \tag{97}$$

where $c = (1/R_C)^{-1}(\widehat{c})$ and $t = (1/R_C)^{-1}(s)$. The rest of the proof for the case (10) is based on the arguments analogous to those in the case (9).

It remains to prove the relation with the class Π when $\eta = \omega$. From Lemma 5, we get $x(s)\operatorname{sgn}x'(s) \in \Pi\left(s\left|\frac{\mathrm{d}x}{\mathrm{d}s}(s)\right|\right)$ no matter whether (9) or (10) holds. Since $s\frac{\mathrm{d}x}{\mathrm{d}s}(s) = \frac{\varphi(t)}{\varphi'(t)}y'(t)$, $\frac{R_D}{R_D'} = r^{\beta-1}R_D$, and $(\frac{1}{R_C})/(\frac{1}{R_C})' = r^{\beta-1}R_C$, we obtain $y\circ(Q^\sigma)^{-1}\operatorname{sgn}y' \in \Pi(r^{\beta-1}Q|y'|)$. □

Proof of Theorem 3. Let us transform Equation (1) into (91) via the relation $y = x\circ(1/R_C)^{-1}$; actually we set $\varphi = R_C$. We have $\widehat{r}(s) = (R_C^{2(1-\alpha)}\circ(1/R_C)^{-1})(s) = s^{2\alpha-2}$ and $|\widehat{p}| = (R_C^2|p|r^{\beta-1})\circ(1/R_C)^{-1} \in \mathcal{RV}(\eta)$. Thus, we can apply Lemma 7. We get $\mathcal{S}^{(91)}\cap\mathcal{NSV} \ne \emptyset$, thus $\mathcal{S}\cap\mathcal{NSV}_{1/R_C} \ne \emptyset$. Take $y \in \mathcal{S}\cap\mathcal{NSV}_{1/R_C}$. Then, $\mathcal{S}^{(91)}\cap\mathcal{NSV}$, and by Lemma 7, $x \in \mathcal{IS}_{BB}^{(91)}\cup\mathcal{DS}_{B0}^{(91)}\cup\mathcal{DS}_{BB}^{(91)}$ when $\widehat{p} < 0$, while $x \in \mathcal{IS}_{B0}^{(91)}\cup\mathcal{IS}_{BB}^{(91)}\cup\mathcal{DS}_{BB}^{(91)}$ when $\widehat{p} > 0$. Recall that $x^{[1]}(s) = y^{[1]}(t)$ where $s = 1/R_C(t)$. Thus, we get (23). Since $x \in \mathcal{IS}_{B0}^{(91)}\cup\mathcal{DS}_{B0}^{(91)}$ satisfies the "circumflex version" of (79), because of (97), the associated $y \in \mathcal{IS}_{B0}\cup\mathcal{DS}_{B0}$ obeys (20). Similarly, (24) follows from the "circumflex version" of (83). □

Proof of Theorem 4. We use transformation of (1) into (91), setting $x(s) = y(t)$, $s = Q^\sigma(t)$, where $y \in \mathcal{S}\cap\mathcal{NRV}_{Q^\sigma}(\vartheta_i)$, $i = 1$ or $i = 2$. We find that $x \in \mathcal{S}^{(91)}\cap\mathcal{NRV}(\Phi^{-1}(\lambda_i))$. Denote $\widehat{L}(s) = s^{-\omega}\widehat{p}(s) - C$. Let (9) hold. Then, $\omega = -\alpha$ and $\lim_{s\to\infty} s^\alpha\widehat{p}(s) = \lim_{t\to\infty} R_D^\alpha(t)p(t)r^{\beta-1}(t) = C$. Moreover, $|\widehat{L}| = |G\circ R_D^{-1}| \in \mathcal{RV}(\tau)$. Thus, we can apply Lemma 8 with

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$\omega = -\alpha$ to (91). Let $b > 0$ be such that $a + 1 = R_D^{-1}(b)$. Using the substitution $v = R_D^{-1}(u)$, we get

$$\int_b^s \frac{\widehat{L}(u)}{u}\,\mathrm{d}s = \int_{a+1}^t \frac{R_D^\alpha(v)p(v)r^{\beta-1}(v) - C}{R_D(v)}\, r^{1-\beta}(v)\,\mathrm{d}v = \int_{a+1}^t G(v)\,\frac{r^{1-\beta}(v)}{R_D(v)}\,\mathrm{d}v, \tag{98}$$

where $t = R_D^{-1}(s)$. Hence, if $\int_{a+1}^\infty |G(t)|r^{1-\beta}(t)/R_D(t)\,\mathrm{d}t = \infty$, then $\int_b^\infty |\widehat{L}(u)|/u\,\mathrm{d}u = \infty$; note that divergence of the integral can happen only when $\tau = 0$, since τ is assumed to be nonpositive. Applying Lemma 8, we get

$$x(s) = s^{\vartheta_i}\exp\left\{-\int_b^s \frac{(1+o(1))\widehat{L}(u)}{(\alpha-1)(\Phi(\vartheta_i) - C/\vartheta_i)u}\,\mathrm{d}u\right\}$$

as $s \to \infty$. In view of (98) and the relation $x = y\circ R_D^{-1}$, we obtain (25). Because of $x = y\circ R_D^{-1}$, $s = R_D(t)$, and (93), the behavior of $L_y^{[R_D]}(t)$ and $y^{[1]}(t)$ is determined by the behavior of $x(s)/s^{\vartheta_i}$ and $x^{[1]}(s)$, respectively. The case when $\int_{a+1}^\infty |G(t)|r^{1-\beta}(t)/R_D(t)\,\mathrm{d}t < \infty$ (with general $\tau \le 0$) can be treated similarly, we use formula (79).

Let (10) hold. Then, $\omega = \alpha - 2$, $\lim_{s\to\infty} s^{2-\alpha}\widehat{p}(s) = \lim_{t\to\infty} R_C^\alpha(t)p(t)r^{\beta-1}(t) = C$, and $|\widehat{L}| = |G\circ(1/R_C)^{-1}| \in \mathcal{RV}(\tau)$. We proceed similarly as in the previous case. The substitution $v = (1/R_C)^{-1}(u)$, which leads to $\mathrm{d}u = (r^{1-\beta}(v)/R_C^2(v))\,\mathrm{d}v$, yields

$$\int_{1/R_C(a)}^s \frac{\widehat{L}(u)}{u}\,\mathrm{d}u = \int_a^t G(v)\,\frac{r^{1-\beta}(v)}{R_C(v)}\,\mathrm{d}v.$$

Applying Lemma 8 with $\omega = \alpha - 2$ to (91), we get the statement.

Assume that $\int_{a+1}^\infty |G(t)|r^{1-\beta}(t)/Q(t)\,\mathrm{d}t < \infty$ (now we work no matter whether (9) or (10) holds). Then, $\lim_{t\to\infty} L_{y_i}^{[Q^\sigma]}(t) = M_i \in (0,\infty)$. Recall that $\widehat{L} = G\circ Q^\sigma$. If $\tau = 0$, then $|M_i - L_{y_i}^{[Q^\sigma]}| \in \mathcal{SV}_{Q^\sigma}$ and (27) follow from $|M_i - L_{x_i}| \in \mathcal{SV}$ and $|\widehat{L}(s)| = o(|M_i - L_{x_i}(s)|)$, respectively (the two latter relations hold by Lemma 8 applied to (91)). Similarly, if $\tau < 0$, then (88) yields (28).

Monotonicity and the limit values of $y_i^{[1]}$ and $L_{y_i}^{[Q^\sigma]}$ also follow from Lemma 8 applied to transformed equation (91).

Let $\tau = 0$. Similarly as in the last paragraph of the proof of Theorem 2, using now Lemma 8, from the relation $L_{x_i}(s)\,\mathrm{sgn}(L_{x_i}(s))' \in \Pi(s|L'_{x_i}(s)|)$, we obtain $(L_{y_i}^{[Q^\sigma]})\circ(Q^\sigma)^{-1}\,\mathrm{sgn}(L_{y_i}^{[Q^\sigma]})' \in \Pi(r^{\beta-1}Q(L_{y_i}^{[Q^\sigma]})')$. □

Proof of Theorem 5. Let $\gamma \ne \alpha - 1$. Then, $\gamma(1-\beta) \ne -1$ and, in view of $r^{1-\beta} \in \mathcal{RV}(\gamma(1-\beta))$ (see Proposition 1-(ii)), the Karamata integration theorem yields

$$Q^\alpha(t)p(t)r^{\beta-1}(t) \sim \left(\frac{1}{(\beta-1)|\gamma-\alpha+1|}\right)^\alpha \frac{t^\alpha p(t)}{r(t)} \sim \left(\frac{1}{(\beta-1)|\gamma-\alpha+1|}\right)^\alpha C_\gamma =: C \in (-\infty, K]$$

as $t \to \infty$. By Theorem 1, $\mathcal{S} \subset \bigcup_{i\in\{1,2\}} \mathcal{NRV}_{Q^\sigma}(\Phi^{-1}(\lambda_i))$, where λ_1, λ_2 are the roots of $|\lambda|^\beta - \sigma\lambda + C/(\alpha-1) = 0$. For $y \in \mathcal{S}$, we thus have $y\circ(Q^\sigma)^{-1} \in \mathcal{NRV}(\Phi^{-1}(\lambda_i))$ where $i = 1$ or $i = 2$. Since $(Q^\sigma)^{-1} \in \mathcal{NRV}(\sigma/((1-\beta)(\gamma+1-\alpha)))$, we have $y \in \mathcal{NRV}(\Phi^{-1}(\lambda_i)\sigma(1-\beta)(\gamma+1-\alpha))$. Observe that $\sigma(1-\beta)(\gamma+1-\alpha) = |\gamma+1-\alpha|/(\alpha-1)$. Set $\eta_i = \lambda_i(|\gamma+1-\alpha|/(\alpha-1))^{\alpha-1}$, $i = 1, 2$. Then, $y \in \mathcal{NRV}(\Phi^{-1}(\eta_i))$. Using the same procedure as in (66), we have

$$|\eta_i|^\beta + \frac{\gamma+1-\alpha}{\alpha-1}\eta + \frac{C_\gamma}{\alpha-1} = \left|\frac{\gamma+1-\alpha}{\alpha-1}\right|^\alpha\left[|\lambda_i|^\beta - \sigma\lambda_i + \frac{C}{\alpha-1}\right] = 0.$$

Thus, η_1, η_2 are the roots of (30). The case $\gamma = \alpha - 1$ is proved in [31]. □

Proof of Corollary 1. By Proposition 1-(ii),(v), $r^{1-\beta} \in \mathcal{RV}(\gamma(1-\beta))$. Since $\gamma \ne \alpha - 1$, we have $\gamma(1-\beta) \ne -1$, so the Karamata integration theorem yields $Q \in \mathcal{RV}((\beta-1)(\alpha-1-\gamma))$. First we show that $|p| \in \mathcal{RV}(\delta)$ and $r \in \mathcal{RV}(\gamma)$ imply $Q^{\alpha+\sigma\omega}|p|r^{\beta-1} \in \mathcal{RV}_{Q^\sigma}(\omega+\tau)$, where

$$\tau = \frac{\delta+\alpha-\gamma}{\sigma(\beta-1)(\alpha-1-\gamma)} = \frac{\delta+\alpha-\gamma}{(\beta-1)|\alpha-1-\gamma|} \le 0;$$

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note that the nonpositivity follows from $\sigma(\alpha-1-\gamma)>0$ (see (11), (12)) and our assumption $\delta+\alpha\le\gamma$. With the help of Proposition 1-(ii),(iii),(v), in view of $(Q^\sigma)^{-1}\in\mathcal{RV}(\sigma/((1-\beta)(\alpha-1-\gamma)))$ recalling that α and β are conjugate number, we have

$$\begin{aligned} Q^{\alpha+\sigma\omega}|p|r^{\beta-1}\circ(Q^\sigma)^{-1} &\in \mathcal{RV}\left(\frac{(\alpha+\sigma\omega)(\beta-1)(\alpha-1-\gamma)+\delta+\gamma(\beta-1)}{\sigma(\beta-1)(\alpha-1-\gamma)}\right)\\ &= \mathcal{RV}\left(\omega+\frac{\alpha(\beta-1)(\alpha-1-\gamma)+\delta+\gamma(\beta-1)}{\sigma(\beta-1)(\alpha-1-\gamma)}\right)=\mathcal{RV}(\omega+\tau). \end{aligned} \tag{99}$$

Next we show that $\lim_{t\to\infty}t^\alpha p(t)/r(t)=0$ implies $\lim_{t\to\infty}Q^\alpha(t)p(t)r^{\beta-1}(t)=0$; note that from Proposition 1-(vi), it follows that the former condition is automatically satisfied when $\delta+\alpha<\gamma$. Theorem 7 yields

$$Q(t)\sim\frac{1}{(\beta-1)|\alpha-1-\gamma|}tr^{1-\beta}(t) \tag{100}$$

as $t\to\infty$. Hence,

$$\begin{aligned} Q^\alpha(t)p(t)r^{\beta-1}(t) &\sim \left(\frac{1}{(\beta-1)|\alpha-1-\gamma|}\right)^\alpha t^\alpha r^{-\beta}(t)p(t)r^{\beta-1}(t)\\ &= \left(\frac{1}{(\beta-1)|\alpha-1-\gamma|}\right)^\alpha\frac{t^\alpha p(t)}{r(t)}\to 0 \end{aligned}$$

as $t\to\infty$. Since the index of regular variation of Q^σ is $(\beta-1)|\alpha-1-\gamma|$, that is, it is nonzero, we have $y\in\mathcal{S}\cap\mathcal{NSV}$ if and only if $y\in\mathcal{S}\cap\mathcal{NSV}_{Q^\sigma}$, and all of the assumptions of Theorem 2 with $\eta=\omega+\tau$ are fulfilled. Further, in view of $(\omega+1)(\beta-1)=\operatorname{sgn}(\gamma+1-\alpha)$, we get $(\eta+1)(\beta-1)|\alpha-1-\gamma|=\gamma+1-\alpha+\tau(\beta-1)|\alpha-1-\gamma|=\gamma+1-\alpha+\delta+\alpha-\gamma=\delta+1$. Hence, because of (100),

$$\Phi^{-1}\left(\frac{Q(t)|p(t)|r^{\beta-2}(t)}{\eta+1}\right)\sim\Phi^{-1}\left(\frac{tp(t)}{(\eta+1)(\beta-1)|\alpha-1-\gamma|r(t)}\right)=\Phi^{-1}\left(\frac{tp(t)}{(\delta+1)r(t)}\right)$$

as $t\to\infty$. Since $\Phi^{-1}(tp(t)/r(t))=\Phi^{-1}(L_p(t)/L_r(t))t^{(\beta-1)(\delta+1-\gamma)}$, applying Theorem 2 (formulas (17) and (20)), we get (31) and (32). Similarly as above, we can show that (21) and (22) imply (33) and (34), respectively. □

Proof of Corollary 2. The statement follows from Theorem 3 with the help of the computations from the proof of Corollary 1. Indeed, with the notation of that theorem and that proof, since (10) holds, we have $\omega=\alpha-2$, $\sigma=-1$, and $Q(t)=R_C(t)\sim tr^{1-\beta}(t)/[(\beta-1)(\gamma-\alpha+1)]$ as $t\to\infty$. Further, in view of (99),

$$(R_C^2|p|r^{\beta-1})\circ(1/R_C)^{-1}=(Q^{\alpha+\sigma\omega}|p|r^{\beta-1})(1/R_C)^{-1}\in\mathcal{RV}(\eta),\ \text{ where } \eta=\alpha-2+\frac{\delta+\alpha-\gamma}{(\beta-1)(\gamma-\alpha+1)}.$$

It is easy to show that $\eta<-1$ (i.e., the condition from Theorem 3 holds) if and only if $\delta<-1$. Consequently, $\mathcal{S}\cap\mathcal{NSV}_{1/R_C}\ne\emptyset$, and since the index of regular variation of $1/R_C$ is positive (because of $\gamma>\alpha-1$), we have $\mathcal{S}\cap\mathcal{SV}\ne\emptyset$. The formulas in (35) easily follow from (24). □

Proof of Theorem 6. Introduce the notation $\xi_i=\Phi^{-1}(\eta_i)$, $i=1,2$. Take $y\in\mathcal{S}\cap\mathcal{NRV}(\xi_i)$, where $i=1$ or $i=2$. Set $w=r\Phi(y'/y)$ and $v=h^\alpha w-Z$, where $Z=rh\Phi(h')$ and $h(t)=t^{\xi_i}$. Then, v satisfies (69) for large t, say $t\ge b\ge a$, $Z(t)=r(t)t^{\xi_i\alpha-\alpha+1}\Phi(\xi_i)$, and

$$\frac{v(t)}{Z(t)}=\frac{t^{\alpha\xi_i}w(t)-Z(t)}{Z(t)}=\frac{\Phi(ty'(t)/y(t))-\Phi(\xi_i)}{\Phi(\xi_i)}\to 0 \tag{101}$$

as $t\to\infty$ since $y\in\mathcal{NRV}(\xi_i)\cap C^1$. Let us write $F(v,Z)=|Z|^\beta F^\#(z)$, where $F^\#(z)=|1+z|^\beta-\beta z-1$, $z=v/Z$. Since $z\to 0$, we may apply the (second-order) Taylor formula to the expression $F^\#$ at the center $z=0$. In view of $F^\#(0)=(F^\#)'(0)=0$, we have $F^\#(z)=(1+o(1))\beta(\beta-1)z^2/2$ as $z\to 0$. Therefore, $F(v(t),Z(t))=(1+o(1))|Z(t)|^{\beta-2}v^2(t)/2$ as

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$t \to \infty$. The modified Riccati equation (69) then takes the form

$$v' + \widetilde{p}(t) + \frac{v^2}{\widetilde{r}(t)} = 0, \tag{102}$$

where $\widetilde{r}(t) = (1 + o(1))2|\xi_i|^{\alpha-2} r(t) t^{\xi_i\alpha-\alpha+2}/\beta$ as $t \to \infty$. In view of

$$\Phi(\xi_i)(\alpha - 1)(\xi_i - 1) = -C_\gamma - \Phi(\xi_i)\gamma, \tag{103}$$

we get

$$\begin{aligned}\widetilde{p}(t) &= t^{\xi_i}\left[r'(t)\Phi(\xi_i)t^{(\alpha-1)(\xi_i-1)} + r(t)\Phi(\xi_i)(\alpha-1)(\xi_i-1)t^{(\alpha-1)((\xi_i-1)-1)} + p(t)t^{\xi_i(\alpha-1)}\right]\\ &= (1+o(1))\frac{\beta}{2}|\xi_i|^{2-\alpha}\frac{\widetilde{r}(t)}{t^2}\left[\Phi(\xi_i)\frac{tr'(t)}{r(t)} + \Phi(\xi_i)(\alpha-1)(\xi_i-1) + \frac{t^\alpha p(t)}{r(t)}\right]\\ &= (1+o(1))\frac{\beta}{2}|\xi_i|^{2-\alpha}\frac{\widetilde{r}(t)}{t^2}G_i(t)\end{aligned}$$

as $t \to \infty$.

Next, we give auxiliary observations for the roots of (14), which then provide desired information on the roots of (30). Put $\delta = \alpha(\vartheta - 1)$, where $\vartheta \in \{\vartheta_1, \vartheta_2\}$, $\vartheta_1 < \vartheta_2$, $\vartheta_i = \Phi^{-1}(\lambda_i)$, $i = 1, 2$, and λ_1, λ_2 satisfy (14) with $C = C_\gamma((\alpha-1)/|\gamma-\alpha+1|)^\alpha$, $\gamma \neq \alpha - 1$. Hence, ϑ_1, ϑ_2 satisfy the equation $|\vartheta|^\alpha - \Phi(\vartheta) + C/(\alpha-1) = 0$. Assume that $C_\gamma > 0$. We have $\delta + 1 - \vartheta + C/\Phi(\vartheta) = (\vartheta - 1)(\alpha - 1) + C/\Phi(\vartheta) = 0$, and so $\delta + 1 = \vartheta - C/\Phi(\vartheta)$. We claim that if $\vartheta = \vartheta_1$, then $\delta < -1$. Indeed, we have $C < K$, where K is defined in (13), which implies $C^\beta(\alpha/(\alpha-1))^\beta < C = C\alpha/(\alpha-1) - C/(\alpha-1)$. Thus, $\lambda_0 := C\alpha/(\alpha-1)$ satisfies $g(\lambda_0) < \lambda_0$, where $g(\lambda) = |\lambda|^\beta + C/(\alpha-1)$. Similarly as in the proof of Lemma 2, inspecting the intersections of the graphs of $\lambda \mapsto g(\lambda)$ and $\lambda \mapsto \lambda$ we find that $\lambda_1 < \lambda_0$. Thus, $|\lambda_1|^\beta = \lambda_1 - C/(\alpha-1) < C$, that is, $|\vartheta_1|^\alpha < C$, which is equivalent to $\delta + 1 = \vartheta_1 - C/\Phi(\vartheta_1) < 0$. Next, we show that if $\vartheta = \vartheta_2$, then $\delta > -1$. Since $\vartheta_2 > (\alpha-1)/\alpha$ (see Lemma 2), we get $\vartheta_2^\alpha > K > C$, which implies $\delta + 1 = \vartheta_2 - C/\Phi(\vartheta_2) > 0$.

Set $\delta_i = \gamma + \alpha(\xi_i - 1)$, $i = 1, 2$. Then, $\widetilde{r} \in \mathcal{RV}(\delta_i + 2)$ and $|\widetilde{p}| \in \mathcal{RV}(\delta_i + \tau_i)$. From (103) we have $\gamma = -\alpha(\xi_i - 1) + \xi_1 - 1 - C_\gamma/\Phi(\xi_i)$, thus $\delta_i = \xi_i - 1 - C_\gamma/\Phi(\xi_i)$, and hence, $\delta_i \lesseqgtr -1$ if and only if $\xi_i - C_\gamma/\Phi(\xi_i) \lesseqgtr 0$. If $C_\gamma \leq 0$, then obviously $\delta_i \lesseqgtr -1$ if and only if $\xi_i \lesseqgtr 0$. Assume that $C_\gamma > 0$ and $\gamma \neq \alpha - 1$. The roots η_i and λ_i of (30) and (14), respectively, are related by $\eta_i = \lambda_i\Phi(\sigma(1-\beta)(\gamma+1-\alpha)) = \lambda_i(|\gamma+1-\alpha|/(\alpha-1))^{\alpha-1}$ (see (66), where $C_\gamma((\alpha-1)/|\gamma-\alpha+1|)^\alpha = C$). We want to show that $\delta_1 < -1$. From the previous paragraph, we know that for $\vartheta_i = \Phi^{-1}(\lambda_i)$, if $i = 1$, then $\vartheta_1 - C/\Phi(\vartheta_1) < 0$. Since $\xi_i = \vartheta_i|\gamma+1-\alpha|/(\alpha-1)$, we get

$$\delta_1 + 1 = \xi_1 - \frac{C_\gamma}{\Phi(\xi_1)} = \vartheta_1\frac{|\gamma+1-\alpha|}{(\alpha-1)} - \left(\frac{\alpha-1}{|\gamma+1-\alpha|}\right)^{\alpha-1}\frac{C_\gamma}{\Phi(\vartheta_1)} = \left(\vartheta_1 - \frac{C}{\Phi(\vartheta_1)}\right)\frac{|\gamma+1-\alpha|}{\alpha-1} < 0.$$

Similarly we prove that $\delta_2 > -1$. Note that $\delta_1 + \tau_1 < -1$ holds because of $\tau_1 \leq 0$ and $\delta_1 < -1$, while $\delta_2 + \tau_2 > -1$ is actually the assumption of the theorem. If $C_\gamma > 0$ and $\gamma = \alpha - 1$, then we come to the equation $|\eta|^\beta + C_\gamma/(\alpha-1) = 0$, which has no real solution.

Equation (102) is in fact the Riccati equation, which is associated to the linear differential equation

$$(\widetilde{r}(t)x')' + \widetilde{p}(t)x = 0 \tag{104}$$

through the substitution $v = \widetilde{r}x'/x$. Since

$$\frac{t^2\widetilde{p}(t)}{\widetilde{r}(t)} \sim \frac{\beta}{2}|\xi_i|^{2-\alpha}G_i(t)$$

and $G_i(t) \to 0$ as $t \to \infty$, we can apply Theorem 5 and Corollary 1 to Equation (104).

First consider the root ξ_1 and assume that $G_1 > 0$. In view of $\delta_1 < -1$ and $1/\widetilde{r} \in \mathcal{RV}(-\delta_1 - 2)$, by Theorem 7, we get $\int_b^\infty 1/\widetilde{r}(s)\,ds = \infty$, and so $\mathcal{S}^{(104)} = \mathcal{IS}^{(104)}$ by Lemma 1. Moreover, Theorem 5 yields $\mathcal{S}^{(104)} \subset \mathcal{NRV}(\zeta_1) \cup \mathcal{NRV}(\zeta_2)$, where ζ_1, ζ_2 are the roots of $\zeta^2 + (\delta_1 + 1)\zeta = 0$, thus either $x \in \mathcal{NSV}$ or $x \in \mathcal{NRV}(-\delta_1 - 1)$. We claim that $x \in \mathcal{NSV}$.

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Indeed, in view of $tx'(t)/x(t) = tv(t)/\widetilde{r}(t)$ and (101),

$$\frac{tx'(t)}{x(t)} = (1+o(1))\frac{\beta}{2}|\xi_1|^{2-\alpha}\left[\Phi\left(\frac{ty'(t)}{y(t)}\right) - \Phi(\xi_1)\right] \to 0 \tag{105}$$

as $t \to \infty$. From Corollary 1, we get

$$\frac{tx'(t)}{x(t)} = -(1+o(1))\frac{t\widetilde{p}(t)}{(\delta_1+\tau_1+1)\widetilde{r}(t)} \tag{106}$$

as $t \to \infty$. Relations (105) and (106) yield

$$\Phi\left(\frac{ty'(t)}{y(t)}\right) - \Phi(\xi_1) = -(1+o(1))\frac{2}{\beta}|\xi_1|^{\alpha-2}\frac{t^2\widetilde{p}(t)}{\widetilde{r}(t)} = -(1+o(1))\frac{G_1(t)}{\delta_1+\tau_1+1} \tag{107}$$

as $t \to \infty$. Set $L_y(t) = t^{-\xi_1}y(t)$. Then, $L_y \in \mathcal{NSV} \cap C^1$ and $ty'(t)/y(t) = \xi_1 + tL_y'(t)/L_y(t)$. Therefore, by the Lagrange mean value theorem,

$$\Phi\left(\frac{ty'(t)}{y(t)}\right) - \Phi(\xi_1) = \Phi\left(\xi_1 + \frac{tL_y'(t)}{L_y(t)}\right) - \Phi(\xi_1) = (\alpha-1)|\nu(t)|^{\alpha-2}(t)\frac{tL_y'(t)}{L_y(t)}, \tag{108}$$

where $\nu(t)$ is between ξ_1 and $\xi_1 + tL_y'(t)/L_y(t)$, and is obviously nonzero for large t. Since $\xi_1 + tL_y'(t)/L_y(t) = (1+o(1))\xi_1$ as $t \to \infty$, we get $\lim_{t\to\infty}\nu(t) = \xi_1$. Hence, (107) and (108) yield

$$\frac{L_y'(t)}{L_y(t)} = (1+o(1))g(t), \quad g(t) := -\frac{|\xi_1|^{2-\alpha}G_1(t)}{(\alpha-1)(\delta_1+\tau_1+1)t}, \tag{109}$$

as $t \to \infty$. Let $\int_a^\infty G_1(t)/t\,\mathrm{d}t = \infty$. Then, τ_1 must be zero, otherwise we would get convergence of the integral because of $G_1 \in \mathcal{RV}(\tau_1)$. Integrating (109) from b to t, we obtain

$$\ln L_y(t) = \ln L_y(b) + \int_b^t (1+o(1))g(s)\,\mathrm{d}s = \int_b^t (1+o(1))g(s)\,\mathrm{d}s = \int_a^t (1+o(1))g(s)\,\mathrm{d}s \tag{110}$$

as $t \to \infty$, where the last two equations follow by the divergence of $\int_a^\infty G_1(t)/t\,\mathrm{d}t$. In view of $\delta_1+\tau_1+1 \le \delta_1+1 < 0$ and $y(t) = t^{\xi_1}L_y(t)$, taking exp in (110), we get (85) with $y(t)/t^{\vartheta_1} \to \infty$. If $\int_a^\infty G_1(t)/t\,\mathrm{d}t < \infty$, then formula (86) follows similarly by integration of (109) from t to ∞. It is clear that $t^{-\vartheta_1}y(t) = L_y(t)$ increases to some positive number M_1. Further, from (109), as $t \to \infty$,

$$L_y'(t) \sim g(t)L_y(t) \sim M_1 g(t) \in \mathcal{RV}(\tau_1 - 1) \tag{111}$$

by Proposition 1-(v). Let $\tau_1 = 0$. Then, $L_y' \in \mathcal{RV}(-1)$. Consequently,

$$L_y(t) \in \Pi(tL_y'(t)) \tag{112}$$

by Proposition 2-(iv) and thus $|M_1 - L_y| \in \mathcal{SV}$ by Proposition 2-(iii). Integrating relation (111) from t to ∞, we obtain $|M_1 - L_y(t)| \sim [M_1/((\alpha-1)|\xi_1|^{\alpha-2}(\delta_1+\tau_1+1))]\int_t^\infty G_1(s)/s\,\mathrm{d}s$ as $t \to \infty$. Since $G_1 \in \mathcal{SV}$ and $G_1(t) = o(\int_t^\infty G_1(s)/s\,\mathrm{d}s)$ by Theorem 7, we get (87). Let $\tau_1 < 0$. Then, by means of Theorem 7, we get $|M_1 - L_y(t)| \sim MG_1(t)/(|\tau_1|(\alpha-1)|\xi_1|^{\alpha-2}|\delta_1+\tau_1+1|)$ as $t \to \infty$.

If we consider the root ξ_2 and assume $G_2 > 0$, then $\delta_2 > -1$. By the assumption $\delta_2 + \tau_2 > -1$ we get $\int_a^\infty 1/\widetilde{r}(s)\,\mathrm{d}s < \infty$ and $\int_a^\infty \widetilde{p}(s)\,\mathrm{d}s = \infty$. Hence, $\mathcal{S}^{(104)} = \mathcal{DS}^{(104)}$ by Lemma 1. Formulas (85), (86), and (87) can be obtained similarly as in the case $\delta_1 < -1$, $G_1 > 0$. Analogously we prove also the cases $G_1 < 0$ and $G_2 < 0$. □

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Proof of Corollary 3. Let us transform (1) into (91) via the relations $s = \varphi(t)$ and $x(s) = y(t)$, where we set $\varphi(t) = \exp t$. Then, $\varphi^{-1} = \ln$ and in view of (92),

$$\widehat{r}(s) = [(r\ln)\Phi(\exp\circ\ln)](s) = s^{\mu+\alpha-1}h(\ln s), \text{ and } \widehat{p}(s) = \left(\frac{p\circ\ln}{\exp\circ\ln}\right)(s) = s^{\xi-1}g(\ln s).$$

Since $h\circ\ln \in \mathcal{NSV}$, we have $\widehat{r} \in \mathcal{NRV}(\widehat{\gamma})$, where $\widehat{\gamma} = \mu+\alpha-1$. Further,

$$\frac{s^\alpha\widehat{p}(s)}{\widehat{r}(s)} = s^{\alpha+\xi-1-\mu-\alpha+1}\frac{g(\ln s)}{h(\ln s)} = \mathrm{e}^{(\xi-\mu)t}\,\frac{g(t)}{h(t)} \to C \in \left(-\infty, \left(\frac{|\mu|}{\alpha}\right)^\alpha\right) = \left(-\infty, \left(\frac{|\alpha-1-\widehat{\gamma}|}{\alpha}\right)^\alpha\right)$$

as $t\to\infty$ (i.e., as $s\to\infty$). We have

$$\frac{s\frac{\mathrm{d}\widehat{r}}{\mathrm{d}s}(s)}{\widehat{r}(s)} = \frac{s\frac{\mathrm{d}}{\mathrm{d}s}(s^\gamma h(\ln s))}{s^\gamma h(\ln s)} = \widehat{\gamma} + \frac{h'(\ln s)}{h(\ln s)}.$$

Therefore,

$$\widehat{G}_i(s) := \frac{s^\alpha\widehat{p}(s)}{\widehat{r}(s)} - C + \eta_i\left(\frac{s^\alpha\widehat{p}(s)}{\widehat{r}(s)} - \widehat{\gamma}\right) = (K_i\circ\ln)(s),$$

and $|\widehat{G}_i| \in \mathcal{RV}(\tau_i)$, $i=1,2$. Hence, all of the assumptions of Theorem 5 and Theorem 6 are fulfilled for Equation (91). Consequently, from Theorem 5 we obtain $\mathcal{S}^{(91)} \subset \mathcal{NRV}(\Phi^{-1}(\eta_1)) \cup \mathcal{NRV}(\Phi^{-1}(\eta_2))$, where $\widehat{\mathcal{S}}$ denotes the set of all eventually positive solutions of (91). This yields $\mathcal{S} \subset \mathcal{NRV}_{\exp}(\Phi^{-1}(\eta_1)) \cup \mathcal{NRV}_{\exp}(\Phi^{-1}(\eta_2))$. Let $y_i \in \mathcal{S}\cap\mathcal{NRV}_{\exp}(\Phi^{-1}(\eta_i))$, $i=1,2$, and set $x_i = y_i\circ\ln$. Then, $x_i \in \mathcal{S}^{(91)}\cap\mathcal{NRV}(\Phi^{-1}(\eta_i))$, and applying Theorem 6, we get that x_i satisfies the "circumflex variant" of (37) or (38) according to whether $\int_{\widehat{a}}^\infty |\widehat{G}_i(u)|/u\,\mathrm{d}u = \infty$ or $\int^\infty |\widehat{G}_i(u)|/u\,\mathrm{d}u < \infty$, respectively. By the substitution $v = \ln u$ in the integral, we have $\int_{\widehat{a}}^s \widehat{G}_i(u)/u\,\mathrm{d}u = \int_a^{\ln s} K_i(v)\,\mathrm{d}v$. Assume that $C\neq 0$ or $C=0$ and η_i is a nonzero root of (39). If $\int_a^\infty |K_i(s)|\,\mathrm{d}s = \infty$, i.e., $\int_{\widehat{a}}^\infty |\widehat{G}_i(u)|/u\,\mathrm{d}u = \infty$, then

$$y_i(\ln s) = x_i(s) = \mathrm{e}^{\Phi^{-1}(\eta_i)\ln s}\exp\left\{-\int_a^{\ln s}(1+o(1))\frac{K_i(v)}{(\alpha-1)(\eta_i - C/\Phi^{-1}(\eta_i))}\,\mathrm{d}v\right\}$$

as $s\to\infty$, from which we easily get (40). Similarly we obtain (41) when $\int_a^\infty |K_i(s)|\,\mathrm{d}s < \infty$. Assume now that $C=0$ and η_i is the zero root. Then, $K_i(t) = \mathrm{e}^{(\xi-\mu)t}\,g(t)/h(t)$, thus $(K_i\circ\ln)(s) = s^{\xi-\mu}g(\ln s)/h(\ln s)$. Since $|K_i\circ\ln| \in \mathcal{RV}(\tau_i)$ and $h\circ\ln\in\mathcal{SV}$, we have $|g\circ\ln|(s) = s^{\mu-\xi}|K_i\circ\ln|(s)(h\circ\ln)(s) \in \mathcal{RV}(\tau_i+\mu-\xi)$. Consequently, $|\widehat{p}| \in \mathcal{RV}(\xi-1+\tau+\mu-\xi) = \mathcal{RV}(\widehat{\delta})$, where $\widehat{\delta} = \tau+\mu-1$. Recall that $r\in\mathcal{RV}(\widehat{\gamma})$, where $\widehat{\gamma} = \mu+\alpha-1$. In the theorem, we assume $\mu<0$ or $\tau+\mu>0$. Thus, let $\mu<0$. Then, $\tau+\mu<0$, and so $\widehat{\delta}<-1$ and $\widehat{\gamma}<\alpha-1$. If $\tau+\mu>0$, then $\mu>0$, and so $\widehat{\delta}>-1$ and $\widehat{\gamma}>\alpha-1$. Moreover, $\widehat{\delta}+\alpha = \tau+\alpha-1+\alpha \le \mu-1+\alpha = \widehat{\gamma}$. Hence, the conditions of Corollary 1 for Equation (91) are fulfilled. Applying that corollary we get that for any $y\in\mathcal{S}\cap\mathcal{NSV}_{\exp}$, taking $x = y\circ\ln$,

$$\begin{aligned} y(\ln s) = x(s) &= \exp\left\{-\int_a^s (1+o(1))\Phi^{-1}\left(\frac{u\widehat{p}(u)}{(\widehat{\delta}+1)\widehat{r}(u)}\right)\mathrm{d}u\right\} \\ &= \exp\left\{-\int_a^{\ln s}(1+o(1))\Phi^{-1}\left(\frac{v^{\xi-\mu}g(v)}{(\tau_i+\mu)h(v)}\right)\mathrm{d}v\right\} \end{aligned}$$

as $s\to\infty$, where we used the substitution $v=\ln u$ in the integral and we assumed $\int_a^\infty |t^{\xi-\mu}g(t)/h(t)|^{\beta-1}\,\mathrm{d}t = \infty$. This leads to (42). Similarly we obtain (43) when $\int_a^\infty |t^{\xi-\mu}g(t)/h(t)|^{\beta-1}\,\mathrm{d}t < \infty$. □

Proof of Corollary 4. It is easy to see that $Q = R_D = \ln^\Theta/\Theta$ with $\Theta>0$ when $\zeta<\alpha-1$ (we set $a=1$), and $Q = R_C = \ln^\Theta/|\Theta|$ with $\Theta<0$ when $\zeta>\alpha-1$. Hence, $Q^\sigma = \ln^{\sigma|\Theta|}/|\Theta|^\sigma$, and so $(Q^\sigma)^{-1}(s) = \exp\{(|\Theta|^\sigma s)^{1/|\Theta|}\}$. We have

$$Q^\alpha(t)p(t)r^{\beta-1}(t) = (1/|\Theta|)^\alpha f_p(t)(\ln t)^{\alpha\Theta+\mu+\zeta(\beta-1)} = (1/|\Theta|)^\alpha f_p(t)(\ln t)^\tau, \tag{113}$$

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where $\tau := \mu + \alpha - \zeta$. If $\tau = 0$, then the expression on the right-hand side tends to zero as $t \to \infty$ by the assumption. If $\tau < 0$, then by the substitution $s = \ln t$, we get $f_p(t)(\ln t)^\tau = g_p(s)s^\tau$, which tends to zero as $s \to \infty$, since $|g_p| \in \mathcal{SV}$. Consequently, for any $\tau \le 0$, $\lim_{t\to\infty} Q^\alpha(t)p(t)r^{\beta-1}(t) = 0$, thus condition (16) (i.e., condition (13) with $C = 0$) is fulfilled. In view of Theorem 1, we get $\mathcal{S} \subset \mathcal{NSV}_{Q^\sigma} \cup \mathcal{NRV}_{Q^\sigma}(\sigma)$. We have $Q^\sigma \in \mathcal{NSV}$ and $\lim_{t\to\infty} t(Q^\sigma(t))'/Q^\sigma(t) = 0$. Hence, in view of the representation formula, see Proposition 3, for $y \in \mathcal{S}$,

$$\frac{ty'(t)}{y(t)} = \frac{t(Q^\sigma(t))'}{Q^\sigma(t)}\psi(t),$$

where $\lim_{t\to\infty}\psi(t) \in \{0, 1\}$. Thus, $\lim_{t\to\infty} ty'(t)/y(t) = 0$ Consequently, $y \in \mathcal{NSV}$ and (44) follows. In fact, the relation $y \in \mathcal{NSV}$ is implied also by Theorem 5 where we have $\gamma = \alpha - 1$ and $C_\gamma = 0$. From $(Q^\sigma)^{-1}(s) = \exp\{(|\Theta|^\sigma s)^{1/|\Theta|}\}$, $\sigma = \operatorname{sgn}\Theta$, $1/\alpha + 1/\beta = 1$, and Proposition 1-(ii),(iii),(v), it is not difficult to deduce that

$$\begin{aligned}
(Q^{\sigma\omega+\alpha}|p|r^{\beta-1})\circ(Q^\sigma)^{-1}(s) &= (Q^{\sigma\omega}|\Theta|^{-\alpha}|f_p|\ln^\tau)\circ(Q^\sigma)^{-1}(s)\\
&= (|\Theta|^{-\alpha-\sigma\omega}|f_p|\ln^{\tau+\Theta\sigma\omega})\circ(Q^\sigma)^{-1}(s)\\
&= |\Theta|^{-\alpha-\sigma\omega}\left|f_p\left(\exp\{(|\Theta|^\sigma s)^{1/|\Theta|}\}\right)\right|(|\Theta|^\sigma s)^{(\tau+\Theta\sigma\omega)/|\Theta|}\\
&= |\Theta|^{\tau/\Theta-\alpha}s^{\omega+\tau/|\Theta|}\left|g_p\left((|\Theta|^\sigma s)^{1/|\Theta|}\right)\right| \in \mathcal{RV}(\omega+\tau/|\Theta|),
\end{aligned}$$

where the last relation follows from $\left|g_p\left((|\Theta|s)^{1/|\Theta|}\right)\right| \in \mathcal{RV}(0/|\Theta|) = \mathcal{SV}$. Thus, condition (15) with $\eta = \omega + \tau/|\Theta|$ is fulfilled. If $\mu < -1$ and $\zeta < \alpha - 1$, we set $\omega = -\alpha$. Recalling that $\tau \le 0$ by the assumption, we get $\eta = \omega + \tau/|\Theta| \le \omega < -1$. If $\mu > -1$ and $\zeta > \alpha - 1$, we set $\omega = \alpha - 2$. We have now $\Theta < 0$, and so

$$|\Theta|(\eta+1) = -\Theta(\eta+1) = -\Theta(\alpha - 1 + \tau/\Theta) = (1-\alpha)\Theta + \tau = \zeta - \alpha + 1 + \mu + \alpha - \zeta = \mu + 1 > 0.$$

Consequently, $\eta > -1$, and therefore $\omega \ge \omega + \tau/|\Theta| = \eta > -1$ since $\tau \le 0$ by the assumption. Hence, all of the assumptions of Theorem 2 are fulfilled. Let $y \in \mathcal{S} \cap \mathcal{NSV}_{Q^\sigma}$. For the expression in the integrands in (17) and (20), we obtain

$$\begin{aligned}
\Phi^{-1}\left(\frac{Q(t)p(t)r^{\beta-2}(t)}{\eta+1}\right) &= \Phi^{-1}\left(\frac{(\ln t)^{\Theta+\mu+\zeta(\beta-2)}f_p(t)t^{(\alpha-1)(\beta-2)-1}}{(\eta+1)|\Theta|}\right)\\
&= \frac{(\ln t)^{(\beta-1)\tau-1}}{t}\Phi^{-1}\left(\frac{f_p(t)}{\mu+1}\right).
\end{aligned} \tag{114}$$

Assume $\int_e^\infty (\ln s)^{(\beta-1)\tau-1}|f_p(s)|^{\beta-1}/s\,\mathrm{d}s = \infty$; note that the divergence of the integral is possible only when $\tau = 0$, which can be seen by using the substitution $t = \ln s$ and the fact that $|g_p| \in \mathcal{SV}$ recalling that $f_p = g_p \circ \ln$. Then, in view of (114), $\int_a^\infty (Q(s)|p(s)|r^{\beta-2}(s))^{\beta-1}\,\mathrm{d}s = \infty$, and therefore, $y \in \mathcal{NSV} \cap \mathcal{S}$ is of the form (17), which can be rewritten as (50). Similarly, if $\int_e^\infty (\ln s)^{(\beta-1)\tau-1}|f_p(s)|^{\beta-1}/s\,\mathrm{d}s < \infty$, then y obeys formula (20), which yields (46).

Next we deal with the solutions $y \in \mathcal{S} \cap \mathcal{NRV}_{Q^\sigma}(\sigma)$. Observe that the assumptions in Theorem 4 in our setting reduce to the conditions (15), (16) discussed in the previous paragraph, and thus are fulfilled. Here, we have $G = f_p/|\Theta|^\alpha$. The nonzero roots (required in the theorem) are $\lambda = \pm 1$ according to whether $\gamma < \alpha - 1$ are $\gamma > \alpha - 1$, thus, the roots can be written as σ. Moreover, $\sigma = \operatorname{sgn}\Theta$. For the expressions in the integrands in (25) and (26), we have

$$\frac{r^{1-\beta}(t)G(t)}{\sigma(\alpha-1)Q(t)} = \frac{f_p(t)(\ln t)^{\tau-1}}{\Phi(\Theta)(\alpha-1)t}.$$

Since $\sigma\lambda = 1$, the expressions $Q^{\sigma\vartheta_i}$ in (25) and (26) are always equal to $Q = \ln^\Theta/|\Theta|$. Consequently, if $\int_e^\infty (\ln s)^{\tau-1}|f_p(s)|/s\,\mathrm{d}s = \infty$ (noting that the divergence is possible only when $\tau = 0$ since $|g_p| \in \mathcal{SV}$), then y is of the form (25), which yields (47); observe that the constant $1/|\Theta|$ which occurred in front of exp can be included in the $(1 + o(1))$ term in the integrand, because of divergence of the integral. Similarly we obtain (48) from (26). □

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Proof of Corollary 5. We proceed similarly as in the proof of Corollary 4. Here, we have $Q = \ln \circ \ln$, and so $Q^{-1} = \exp \circ \exp$. Note that (9) holds. From the assumptions and since $(\alpha - 1)(\beta - 1) = 1$, we get

$$(|L_p|L_r^{\beta-1}) \circ \exp \circ \exp = |f_p \circ \exp \circ \exp| = |h_p| \in \mathcal{RV}(-\alpha).$$

Moreover,

$$Q^{\alpha}(t)p(t)r^{\beta-1}(t) = (\ln \ln t)^{\alpha} f_p(t) \to 0$$

as $t \to \infty$. Hence, the assumptions of Theorem 1, Theorem 2, and Theorem 4 are satisfied. Thus, we get $\mathcal{S} \subset \mathcal{NSV}_{\ln \circ \ln} \cup \mathcal{NRV}_{\ln \circ \ln}(1)$. It is easy to see that $\mathcal{NSV}_{\ln \circ \ln} \cup \mathcal{NRV}_{\ln \circ \ln}(1) \subset \mathcal{NSV}$ and so (49) follows. Noting that $\omega = -\alpha$, for the expressions in the integrands in (17) and (20), we obtain

$$\Phi^{-1}\left(\frac{Q(t)p(t)r^{\beta-1}(t)}{\omega + 1}\right) r^{1-\beta}(t) = -\frac{1}{t \ln t} \Phi^{-1}\left(\frac{f_p(t) \ln \ln t}{\alpha - 1}\right).$$

Consequently, for $y \in \mathcal{NSV}_{\ln \circ \ln}$, formulas (17), (20), and (21) yield (50), (51), and (52), respectively. For the expressions in the integrands in (25) and (26), we have

$$\frac{r^{1-\beta}(t)G(t)}{\sigma(\alpha - 1)Q(t)} = \frac{(\ln \ln t)^{\alpha-1}}{(\alpha - 1)t \ln t} f_p(t).$$

Now it is easy to see that (25), (26), and (27) yield (53), (54), and (55), respectively. □

Proof of Corollary 6. Take $y \in S^{(}56)$ and note that nonoscillation of (56) will follow from the following observations. Set $x(s) = y(t)$, $s = \mathrm{e}^t$, and $t_0 \geq 0$. Then, (56) is transformed into the equation

$$s(s^{\alpha-1}\Phi(x'))' + \widetilde{a}(s)s^{\alpha-1}\Phi(x') + \widetilde{b}(s)\Phi(x) = 0,$$

where $\widetilde{a} = a \circ \ln$, $\widetilde{b} = b \circ \ln$, and by $'$ we mean $\frac{\mathrm{d}}{\mathrm{d}s}$. The interval $[t_0, \infty)$ is transformed into $[s_0, \infty)$, where $s_0 > 0$. Differentiating out the first term and dividing the transformed equation by s^{α}, we get

$$\mathcal{L}[x] := (\Phi(x'))' + \frac{\alpha - 1 + \widetilde{a}(s)}{s}\Phi(x') + \frac{\widetilde{b}(s)}{s^{\alpha}}\Phi(x) = 0.$$

Set

$$r(s) = \exp\left\{\int_{s_0}^{s} \frac{\alpha - 1 + \widetilde{a}(z)}{z}\,\mathrm{d}z\right\}, \quad p(s) = \frac{\widetilde{b}(s)r(s)}{s^{\alpha}}.$$

Then,

$$r(s)\mathcal{L}[x] = r(s)\left((\Phi(x'))' + \frac{r'(s)}{r(s)}\Phi(x') + \frac{p(s)}{r(s)}\Phi(x)\right)$$
$$= (r(s)\Phi(x'))' + p(s)\Phi(x).$$

Hence, $x \in S^{(115)}$, where Equation (115) is given by

$$(r(s)\Phi(x'))' + p(s)\Phi(x) = 0. \tag{115}$$

Since $sr'(s)/r(s) = \alpha - 1 + \widetilde{a}(s) \to \alpha - 1 + A$ as $s \to \infty$, we have $r \in \mathcal{NRV}(\alpha - 1 + A) \cap C^1$. Moreover, $\lim_{s\to\infty} s^{\alpha}p(s)/r(s) = \lim_{s\to\infty} \widetilde{b}(s) = B$. Thus, the assumptions of Theorem 5, where $\gamma = \alpha - 1 + A$ are fulfilled for (115). Note that for the roots λ_1, λ_2 of (57) it holds that if $\eta_i = \Phi(\lambda_i)$, $i = 1, 2$, then η_i are the roots of (30), where for the coefficients of that algebraic equation we have $(\alpha - 1 - \gamma)/(\alpha - 1) = -A/\alpha - 1$ and $C_{\gamma} = B$. Hence, $S^{(115)} \subset \bigcup_{i\in\{1,2\}} \mathcal{NRV}(\lambda_i)$,

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which implies, in view of $y \circ \ln = x$, $S^{(56)} \subset \bigcup_{i\in\{1,2\}} \mathcal{NRV}_{\exp}(\lambda_i)$. Hence,

$$\lim_{t\to\infty} \frac{y'(t)}{y(t)} = \lim_{s\to\infty} \frac{s\frac{\mathrm{d}x}{\mathrm{d}s}(s)}{x(s)} \in \{\lambda_1, \lambda_2\}.$$

Let us consider a nonzero root η_i (thus a nonzero root λ_i). We have

$$\begin{aligned} H_i(t) &= \widetilde{b}(s) - B + \eta_i(\alpha - 1 + \widetilde{a}(s) - \alpha + 1 - A) \\ &= \frac{s^\alpha p(s)}{r(s)} - B + \eta_i\left(\frac{sr'(s)}{r(s)} - (\alpha - 1 + A)\right). \end{aligned}$$

The last expression is in fact $G_i(s)$ (G_i is defined by (36)) where $C_\gamma = B$ and $\gamma = \alpha - 1 + A$. Since $|G_i| = |H_i \circ \ln| \in \mathcal{RV}(\tau_i)$, we see that the assumptions of Theorem 6 are fulfilled. The substitution yields

$$\int_{s_0}^{s} \frac{G_i(z)}{z}\,\mathrm{d}z = \int_{\ln s_0}^{\ln s} H_i(u)\,\mathrm{d}u,$$

and thus from the asymptotic formulas in Theorem 6, we get (59) and (60). Note that $\int_a^\infty |H_i(s)|\,\mathrm{d}s$ can diverge only when $\tau_i = 0$. Indeed, we know that $\int_a^\infty |H_i(s)|\,\mathrm{d}s = \infty$ if and only if $\int_a^\infty |G_i(s)|/s\,\mathrm{d}s = \infty$. If τ_i were negative, then $|G_i(t)|/t$ would be regularly varying of the index less than -1, and so the integral would be convergent by the Karamata theorem. Now consider the case where $\eta_i = 0$ for $i = 1$ or $i = 2$ (which can happen only when $B = 0$). Then,

$$\frac{s^\alpha |p(s)|}{r(s)} = |\widetilde{b}(s)| = |H_i(t)| = b(t) \in \mathcal{RV}_{\exp}(\tau_0).$$

Hence, $|p(s)| = |\widetilde{b}(s)| r(s)/s^\alpha \in \mathcal{RV}(A - 1 + \tau_0)$. We assume either $A < 0$ or $A + \tau_0 > 0$. If $A < 0$, then $A + \tau_0 < 0$. On the other hand, if $A + \tau_0 > 0$, then $A > 0$. In particular, we have $\delta := A - 1 + \tau_0 \lneqq -1$ and $\gamma := \alpha - 1 + A \lneqq \alpha - 1$. For the integrals in formulas (31) and (32) we have

$$\int_{s_0}^{s} z^{(1+\delta-\gamma)(\beta-1)} \Phi^{-1}\left(\frac{L_p(z)}{L_r(z)}\right)\mathrm{d}z = \int_{s_0}^{s} \Phi^{-1}\left(\frac{zp(z)}{r(z)}\right)\mathrm{d}z = \int_{s_0}^{s} \frac{\Phi^{-1}(\widetilde{b}(z))}{z}\,\mathrm{d}z = \int_{\ln s_0}^{\ln s} \Phi^{-1}(b(u))\,\mathrm{d}u.$$

Now it is easy to see how Corollary 1 yields (61) and (62). □

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ORCID

Pavel Řehák https://orcid.org/0000-0002-2452-2204

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