



Periodic solutions to history-dependent differential hemivariational inequalities with applications

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ABSTRACT

In this article, we investigate a hybrid model combined by a parabolic differential equation and a parabolic hemivariational inequality (so-called differential hemivariational inequality of parabolic–parabolic type) in general infinite dimensional spaces which includes the history-dependent operator. The solvability of initial value problems as well as the periodic problems of the hemivariational inequality and the differential hemivariational inequality have been proved. In application, we study a contact problem with normal compliance driven by a history-dependent dynamical system.

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1. Introduction

Variational inequalities constitute one of the most vibrant branches of applied mathematics that have been extended in a multitude of directions. While the classical variational inequalities deal with the models with monotone process, the hemivariational inequalities involve nonmonotone relations described by nonconvex subdifferential, which have evolved significantly in relation to multivalued, nonmonotone, nonconvex optimization, and structural analysis (see, e.g. [1]). In the early 1980s, Panagiotopoulos first introduced hemivariational inequalities in [2,3], which related to Clarke's development of the generalized gradient of a locally Lipschitz function. Hemivariational inequalities have been shown to be useful in a wide range of phenomena, including nonsmooth mechanics, engineering, physics as well as economics. As a result, a large number of applications lead to mathematical models expressed in terms of hemivariational

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inequalities. The theory of nonlinear inclusions and hemivariational inequalities with emphasis on contact mechanics can be found in the monographs [4], in the extensive works [5–10] and in references cited therein.

The notion and study of differential hemivariational inequalities were first introduced in [11], for a new type of hemivariational inequalities containing derivatives. This problem has progressed in a number of ways recently, see [12–15]. Liu, Zeng and Motreanu initially proposed the idea of the differential hemivariational inequality (DVHI) in [16], which is formulated by a hemivariational inequality and a nonlinear evolution equation. A feedback iterative approach and backward Euler difference scheme were then introduced by Migórski and Zeng [17] to investigate an abstract DHVI in the context of an evolution triple of spaces for hemivariational inequalities. DHVIs can also serve as a powerful and useful mathematical tool to model-reality problems like frictional quasistatic contact problem for viscoelastic materials with adhesion [18,19] or dynamic frictionless viscoplastic contact problem [20,21].

To precisely state the problem under consideration, let $(X, \|\cdot\|_X)$ and $(E, \|\cdot\|_E)$ be two Banach spaces. Let $(U, \|\cdot\|_U)$ be another Banach space, let U^* refer to the dual of U with the corresponding dual pairing $\langle \cdot, \cdot \rangle$. Suppose that H is a Hilbert space such that the embedding $U \subset H \subset U^*$ forms an evolution triple of spaces. We consider a system which consists of a historical-dependent evolution equation problem combined with a parabolic hemivariational inequality as follows:

$$x'(t) = Ax(t) + (Rx)(t) + F(t, x(t), u(t)), \quad \text{a.e. } t \in I := [0, b], \quad (1.1)$$

$$\langle u'(t), v \rangle + \langle B(t, u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \geq \langle g(t, x(t), u(t)), v \rangle, \quad (1.2)$$

$$\text{for a.e. } t \in I, \text{ for all } v \in U,$$

$$x(0) = x(b), \quad u(0) = u(b), \quad (1.3)$$

where $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on X . The history-dependent function $R : L^2(0, b; X) \rightarrow L^2(0, b; X)$ and the functions $F : I \times X \times H \rightarrow X$, $M : H \rightarrow E$, $B : I \times U \rightarrow U^*$, $g : I \times X \times H \rightarrow H$ are given which will be specified in the next sections. For $(t, x) \in I \times X$, the notation $J^0(t, x, \cdot; \cdot)$ represents the generalized directional derivative of the locally Lipschitz mapping $J : I \times X \times E \rightarrow \mathbb{R}$.

Let us briefly discuss the above system. When $R = 0$, (1.1)–(1.2) was studied in [22]. The authors discussed the solvability and properties of solution sets, then they treated the long time behaviour of solutions by proving that there are pullback attractors for multivalued processes associated with DHVIs. The history-dependent operators appearing in (1.1) allow us to express the dynamical systems combining with the previous state. Results on history-dependent differential variational–hemivariational inequalities were investigated in [23], and several kinds of evolutionary differential variational–hemivariational inequalities were discussed in [15]. For further application of differential hemivariational inequalities involving history-dependent terms, we refer to [24].

It turns out that studying the periodic problem plays a crucial role in many applications and various processes coming from reality-phenomena in various mechanical models. In theoretical problems, periodicity is a common important feature of many dynamical systems. The existence of periodic solutions of dynamical systems, differential variational inequalities and some related forms has been studied in [25–29]. To the best of the authors knowledge, up to now, in the literature, there still has not been any study on periodic solutions for the differential hemivariational inequalities (1.1)–(1.2), and that is the main motivation of our paper.

The approach used in this work is in the spirit of techniques in variational analysis (surjectivity lemma), in some fixed point arguments and in the theory of measure of noncompactness (which is very useful tools and widely used in fixed point theory). The results on the existence of periodic solution for parabolic hemivariational inequalities as well as for differential hemivariational inequalities are established.

This article is structured as follows. First, we recall some basic definitions and fundamental properties of generalized Clarke directional derivative and generalized gradient of a nonlocal Lipschitz functions, the variational-constant formula for the solutions to Cauchy problem via the theory of semigroup in Section 2. In Section 3, we state and prove the results on solvability to initial value problem and periodic problem for the parabolic hemivariational inequality. Section 4 shows the existence of mild solutions and Section 5 provides results on periodic solutions for (1.1)–(1.2). In the last section, we discuss a contact problem in solid mechanics under the impact of temperature.

2. Preliminaries

2.1. Generalized directional derivative and generalized gradient

Now we recall the notations and some facts about generalized gradient (see, e.g. [1]). It is known that the generalized gradient is a replacement for the derivative. It can be defined for a very general class of functions. Generalized gradient has an important role as the essential need to grapple with non-smoothness as well as with absence of convexity appearing in many reality theories and applications such as optimization and control problems, the viscosity solutions and optimal design, variational methods, etc.

For a Banach space E , a function $J : E \rightarrow \mathbb{R}$ is called to be locally Lipschitz, that means, for each $u \in E$, there exist a neighbourhood $N(u)$ of u in E and a constant $L_u > 0$ such that

$$|J(u_1) - J(u_2)| \leq L_u \|u_1 - u_2\|_E, \quad \forall u_1, u_2 \in N(u).$$

Let $J^0(u; v)$ be the generalized (Clarke) directional derivative of J in the direction $v \in E$ at the point $u \in E$ defined by

$$J^0(u; v) = \limsup_{\lambda \rightarrow 0^+, z \rightarrow u} \frac{J(z + \lambda v) - J(z)}{\lambda}.$$

The generalized gradient of $J : E \rightarrow \mathbb{R}$ at $u \in E$, denoted $\partial J(u)$, is a subset of E^* given by

$$\partial J(u) = \{\gamma \in E^* \mid J^0(u; v) \geq \langle \gamma, v \rangle, \forall v \in E\}.$$

The following proposition formulates some results concerning the characteristics of the generalized gradients and the Clarke directional derivatives for locally Lipschitz functions.

Proposition 2.1. *Let $J : E \rightarrow \mathbb{R}$ be a locally Lipschitz function. Then, the following statements hold:*

- (i) *For each $u \in E$, the function $E \ni v \mapsto J^0(u; v) \in \mathbb{R}$ is positively homogeneous, subadditive, and the estimate*

$$|J^0(u; v)| \leq L_u \|v\|_E, \quad \forall v \in E$$

holds, where $L_u > 0$ stands for the Lipschitz constant of J near u .

- (ii) *The function $E \times E \ni (u, v) \mapsto J^0(u; v) \in \mathbb{R}$ is upper semicontinuous.*

- (iii) *For any $v \in E$, one has $J^0(u; v) = \max\{\langle u^*, v \rangle \mid u^* \in \partial J(u)\}$.*

2.2. Monotone operator and surjectivity result

Let U be a Banach space with its dual U^* . We introduce the following definition for a single-valued mapping from U to U^* .

Definition 2.1. A single-valued operator $B : U \rightarrow U^*$ is called:

(i) monotone if

$$\langle Bu - Bv, u - v \rangle \geq 0,$$

for all $u, v \in U$,

(ii) strongly monotone if there exists a constant $M_B > 0$ such that

$$\langle Bu - Bv, u - v \rangle \geq M_B \|u - v\|^2,$$

for all $u, v \in U$,

(iii) pseudomonotone if B is a bounded operator; moreover, for all sequence $\{u_n\} \subset U$ with $u_n \rightarrow u$ weakly in U and $\limsup \langle Bu_n, u_n - v \rangle \leq 0, \forall v \in U$, then

$$\langle Bu, u - v \rangle \leq \liminf \langle Bu_n, u_n - v \rangle,$$

(iv) demicontinuous, if the mapping $u \mapsto \langle Bu, v \rangle$ is continuous for all $v \in U$, i.e., B is continuous as a mapping from U to U^* with weak topology in U^* .

Definition 2.2. The multimap $B : U \rightarrow \mathcal{P}(U^*)$ is called pseudomonotone associated to $D(\mathcal{L})$ (so-called $\mathcal{L}$ -pseudomonotone), where the operator $\mathcal{L} : D(\mathcal{L}) \subset U \rightarrow U^*$ is linear, closed, densely defined, and maximal monotone, if

- (i) for each $u \in U$, the set Bu is nonempty, closed, bounded and convex in U^* ;
- (ii) B is an upper semicontinuous mapping from any finite-dimensional subspace of U to U^* endowed with the weak topology;
- (iii) for any sequence $\{u_n\} \subset D(\mathcal{L})$ and any sequence $\{u_n^*\} \subset U^*$ with the followings:

$$\begin{cases} u_n \rightarrow u \text{ weakly in } U, \\ \mathcal{L}u_n \rightarrow \mathcal{L}u \text{ weakly in } U^*, \\ u_n^* \in Bu_n \text{ for all } n \text{ in } \mathbb{N}, \\ u_n^* \rightarrow u^* \text{ weakly in } U^*, \\ \limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0, \end{cases}$$

we have $u^* \in Bu$ and $\lim_{n \rightarrow \infty} \langle u_n^*, u_n \rangle = \langle u^*, u \rangle$.

We now state the surjectivity result that will be used in the next section.

Lemma 2.2. Suppose that U is a reflexive, strictly convex Banach space, $L : D(L) \subset U \rightarrow U^*$ is a densely defined linear maximal monotone operator and $T : U \rightarrow 2^{U^*}$ is bounded, coercive and L -generalized pseudomonotone, then

$$\text{Range}(L + T) = U^*.$$

The proof of Lemma 2.2 can be found in [30, Corollary 1, p. 610] for T single-valued and in [16] for T multivalued.

2.3. The Cauchy problem generated by the (unbounded) linear generator

Let X be a Banach space, a family of operators $\{S(t) : X \rightarrow X\}_{t \geq 0}$ is said to be a one-parameter semigroup if the following statements hold:

- (i) $S(0) = Id$ where Id is the identity operator on X ;
- (ii) $S(t + s) = S(t)S(s)$ for all $t, s \geq 0$.

The semigroup $\{S(t)\}_{t \geq 0}$ is a C_0 -semigroup (strongly continuous) on X if for each $x \in X$, we have $\lim_{t \rightarrow 0^+} \|S(t)x - x\|_X = 0$. The semigroup $\{S(t)\}_{t \geq 0}$ is called to be:

(a) exponentially stable if there are positive numbers P, γ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Pe^{-\gamma t}, \text{ for all } t \geq 0;$$

(b) compact if for each $t > 0$, $S(t)$ is a compact operator.

The operator $A : D(A) \subseteq X \rightarrow X$ is called to be a generator of the strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ on X if

$$Ax := \lim_{h \downarrow 0} \frac{1}{h} (S(h)x - x)$$

which its domain

$$D(A) := \{x \in X : \text{the map } t \mapsto S(t)x \text{ is differentiable}\}.$$

Now we examine the linear Cauchy problem as follows

$$\begin{cases} x'(t) &= Ax(t) + g(t), \\ x(0) &= x_0. \end{cases} \quad (2.1)$$

Then the variational-constant formula for the solutions to (2.1) is given by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)g(s)ds.$$

The well-known Schauder fixed point theorem which the original proof can be found in [31] reads as follows.

Lemma 2.3. *Let D be a nonempty compact convex subset in a Banach space E . Then the continuous mapping $\mathcal{F} : D \rightarrow D$ has a fixed point.*

2.4. Measure of non-compactness

In this subsection, we briefly review basic notation and results of the theory of measures of noncompactness in Banach spaces which are needed in the paper. For more details, we refer to the monographs [32].

Let E be a Banach space and we define:

$$\mathcal{B}(E) = \{B \in \mathcal{P}(E) : B \text{ is bounded}\}.$$

We first recall definitions and properties of measures of noncompactness.

Definition 2.3. A function $\beta : \mathcal{B}(E) \rightarrow \mathbb{R}^+$ is called a measure of noncompactness (MNC) in E if

$$\beta(\overline{\text{co}}\Omega) = \beta(\Omega) \text{ for every } \Omega \in \mathcal{B}(E),$$

where $\overline{\text{co}}\Omega$ is the closure of the convex hull of Ω . An MNC β is called

- (i) monotone if $\Omega_0, \Omega_1 \in \mathcal{B}(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
- (ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for any $a \in E$, $\Omega \in \mathcal{B}(E)$;
- (iii) invariant with respect to union with compact set if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset E$ and $\Omega \in \mathcal{B}(E)$;
- (iv) algebraically semi-additive if for any $\Omega_0, \Omega_1 \in \mathcal{B}(E)$ we have $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$;
- (v) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω .

Well-known examples of measures of noncompactness satisfying all of the above properties are the Hausdorff MNC and the Kuratowski MNC, defined by

$$\begin{aligned}\chi(\Omega) &= \inf\{\epsilon : \Omega \text{ has a finite } \epsilon - \text{net}\}, \\ \alpha(\Omega) &= \inf\{\epsilon : \Omega \text{ admits a partition into a finite number} \\ &\quad \text{of sets whose diameters are less than } \epsilon\},\end{aligned}$$

respectively.

We recall the following statement (see [33, Proposition 2.5]) for the estimates related to the Hausdorff MNC.

Proposition 2.4. *Let $D \subset L^1(0, T; E)$ such that*

- (1) $\|\xi(t)\| \leq \nu(t)$, for all $\xi \in D$ and for a.e. $t \in [0, T]$,
- (2) $\chi(D(t)) \leq q(t)$ for a.e. $t \in [0, T]$,

where $\nu, q \in L^1(0, T; \mathbb{R})$. Then

$$\chi\left(\int_0^t D(s)ds\right) \leq 4 \int_0^t q(s)ds,$$

here $\int_0^t D(s)ds = \{\int_0^t \xi(s)ds : \xi \in D\}$.

3. The initial value problem and periodic problem for the parabolic hemivariational inequality

In this section, we study the existence and uniqueness of the solutions and the continuous dependence of solutions to abstract hemivariational inequality problem on the time interval $[0, b]$ with given initial data. To get this purpose, we establish the following conditions imposed on the functions B , J , M , g as follows:

$H(B)$ $B : [0, b] \times U \rightarrow U^*$ satisfies:

- (1) $B(\cdot, u)$ is measurable on $[0, b]$ for all $u \in U$;
- (2) $B(t, \cdot)$ is pseudomonotone and demicontinuous for a.e. $t \in [0, b]$;
- (3) there are a positive number α_B and a positive function $\eta_B \in L^2([0, b]; \mathbb{R}_+)$ such that

$$\|B(t, v)\|_{U^*} \leq \eta_B(t) + \alpha_B \|v\|_U, \quad \text{for all } t \in [0, b] \text{ and } v \in U;$$

- (4) $B(t, \cdot)$ is strongly monotone for a.e. $t \in [0, b]$, that is, there exists a constant $\beta_B > 0$ such that

$$\langle B(t, u) - B(t, v), u - v \rangle \geq \beta_B \|u - v\|_U^2,$$

for all $t \in [0, b]$ and for all $u, v \in U$.

$H(J)$ $J : [0, b] \times X \times E \rightarrow \mathbb{R}$ is such that:

- (1) $J(\cdot, x, v)$ is measurable for all $x \in X, v \in E$ and $J(t, \cdot, \cdot)$ is continuous in $X \times E$;
- (2) $J(t, x, \cdot)$ is locally Lipschitz continuous for all $(t, x) \in [0, b] \times X$;
- (3) there exist $\alpha_J > 0$ and $\eta_J \in L^2(0, b; \mathbb{R}_+)$ satisfying

$$\|\partial J(t, x, v)\|_{E^*} \leq \eta_J(t) + \alpha_J (\|x\|_X + \|v\|_E),$$

for all $(t, x, v) \in [0, b] \times X \times E$.

- (4) there exists a constant $\beta_J \geq 0$ such that

$$\langle \xi_1 - \xi_2, v_1 - v_2 \rangle \geq -\beta_J (\|v_1 - v_2\|_E^2 + \|x_1 - x_2\|_X \|v_1 - v_2\|_E),$$

for every $\xi_1 \in \partial J(t, x_1, v_1), \xi_2 \in \partial J(t, x_2, v_2), x_1, x_2 \in X, v_1, v_2 \in E$.

$H(M)$ M is a bounded linear continuous operator from H to E ($M \in \mathcal{L}(H, E)$), we use the notation $\|M\|$ referring to the norm of the operator M .

$H(g)$ $g : [0, b] \times X \times H \rightarrow H$ is measurable w.r.t. the first variable, continuous w.r.t. the second and third variables. Moreover, there is a positive constant α_g satisfying

$$|g(t, x_1, v_1) - g(t, x_2, v_2)|_H \leq \alpha_g(|v_1 - v_2|_H + \|x_1 - x_2\|_X),$$

for all $x_i \in X, v_i \in H$ ($i = 1, 2$), and $\|g(\cdot, 0_X, 0_H)\|_H := \gamma_g(\cdot) \in L^2(0, b; \mathbb{R}_+)$, where 0_X and 0_H stand for the zero elements of X and H , respectively.

$H(0)$ $\beta_B > \max\{(\alpha_J\|M\| + \alpha_g)\ell^2, (\beta_J\|M\|^2 + \alpha_g)\ell^2\}$, where ℓ stands for the embedding constant from U into H .

Let $x \in C([0, b]; X)$, $u_0 \in U$ be given, we consider the following hemivariational inequality:

$$\begin{aligned} \langle u'(t), v \rangle + \langle B(t, u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \\ \geq \langle g(t, x(t), u(t)), v \rangle, \quad \forall v \in U, \text{ for a.e } t \in [0, b], \end{aligned} \quad (3.1)$$

$$u(0) = u_0. \quad (3.2)$$

We have the existence result for (3.1)–(3.2).

Lemma 3.1. *Suppose that the hypotheses $H(B), H(J), H(M), H(g)$ and $H(0)$ hold. For each $x \in C([0, b]; X)$, the problem (3.1) and (3.2) admits at least one solution. The solution is unique. Moreover, there are two positive numbers $\beta, \rho > 0$ with $\beta = \beta_B - (\beta_J\|M\|^2 + \alpha_g)\ell^2$ and $\rho = \beta_J\|M\| + \alpha_g$ such that*

$$|u_1(t) - u_2(t)|_H \leq e^{-\beta t} |u_1(0) - u_2(0)|_H + \rho \int_0^t e^{-\beta(t-s)} \|x_1(s) - x_2(s)\|_X ds, \quad (3.3)$$

where u_1 and u_2 are the unique solutions to the problem (3.1)–(3.2) with respect to the given input continuous functions x_1 and x_2 on the time interval $[0, b]$.

Proof. We divide the proof into several steps as follows.

Step 1: The existence of a solution with the initial condition.

This statement can be proved by using the surjectivity result. We introduce the following functional spaces:

$$\begin{aligned} \mathcal{U} &:= L^2(0, b; U), \\ \mathcal{H} &:= L^2(0, b; H) = \mathcal{H}^*, \\ \mathcal{U}^* &:= L^2(0, b; U^*), \\ \mathcal{W} &:= \{w \in \mathcal{U} | w' \in \mathcal{U}^*\}, \end{aligned}$$

then we get the following inclusions

$$\mathcal{W} \subset \mathcal{U} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{U}^*.$$

We denote $\langle \langle \cdot, \cdot \rangle \rangle$ the pairing between $\mathcal{U}$ and $\mathcal{U}^*$. Let $\mathcal{B} : \mathcal{U} \rightarrow \mathcal{U}^*$ defined by:

$$\begin{aligned} \mathcal{B}(u)(\cdot) &= B(t, u(\cdot) + u_0) - g(\cdot, x(\cdot), u(\cdot) + u_0), \\ \langle \mathcal{B}(u)(t), v \rangle &= \langle B(t, u(t) + u_0) - g(t, x(t), u(t) + u_0), v \rangle, \end{aligned}$$

The function $\mathcal{J}$ can be stated by:

$$\mathcal{J} : \mathcal{U} \rightarrow 2^{\mathcal{U}^*},$$

$$\mathcal{J}(u) = \{v \in \mathcal{H} : v(t) \in M^* \partial J(t, x(t), Mu(t) + Mu_0)\}.$$

Put

$$\begin{aligned} \mathcal{T} : \mathcal{U} &\rightarrow 2^{\mathcal{U}^*} \\ \mathcal{T}(u) &= \mathcal{B}u + \mathcal{J}u, \\ \mathcal{L} : D(\mathcal{L}) \subset \mathcal{U} &\rightarrow \mathcal{U}^*, \\ \mathcal{L}u &= u', \\ D(\mathcal{L}) &= \{u \in \mathcal{U} : u' \in \mathcal{U}^* \text{ and } u(0) = 0\} \subset \mathcal{W}, \end{aligned}$$

Then the system (3.1) and (3.2) can be read as:

$$\mathcal{L}u + \mathcal{T}u \ni 0.$$

We have the basic properties of $\mathcal{T}$ listed below:

(i) $\mathcal{T}$ is bounded;

This is evident based on the boundedness of the operator $\mathcal{B} : \mathcal{U} \rightarrow \mathcal{U}^*$ and the multivalued mapping $\mathcal{J}$ from $\mathcal{U}$ to $2^{\mathcal{U}^*}$.

(ii) $\mathcal{T}$ is coercive;

Let $u \in \mathcal{U}$ and $u^* \in \mathcal{J}(u)$, we get

$$\begin{aligned} -\langle u^*, u \rangle &\leq \mathcal{J}^\circ(u; -u) \\ &\leq \max\{\langle w, -u \rangle | w \in \partial \mathcal{J}(u)\} \\ &\leq \max \left\{ \int_0^b \langle w(t), -u(t) \rangle_{\mathcal{U}^*, \mathcal{U}} dt | w \in \partial \mathcal{J}(u) \right\} \\ &\leq \int_0^b \|u(t)\|_H \|M^* (\eta_J(t) + \alpha_J(\|x(t)\|_X + \|u(t)\|_H))\| dt \\ &\leq \|M\| \ell^2 \alpha_J \|u\|_{\mathcal{U}}^2 + \|M\| (\alpha_J \|x\|_{C([0,b];X)} + \|\eta_J\|_{L^2(0,b)}) \|u\|_{\mathcal{H}}. \end{aligned}$$

Hence, $\mathcal{J}$ is subcoercive. Moreover, by using H(B) and H(g), one has

$$\langle \mathcal{B}(u) - \mathcal{B}v, u - v \rangle \geq (\beta_B - \alpha_g \ell^2) \|u - v\|_{\mathcal{U}}, \quad \forall u, v \in \mathcal{U}.$$

Then, it follows from the condition $\beta_B > (\alpha_g + \|M\| \alpha_J) \ell^2$ that $\mathcal{B} + \mathcal{J}$ is coercive.

(iii) $\mathcal{T}$ is a $\mathcal{L}$ -generalized pseudomonotone operator.

Because $\mathcal{J}$ has nonempty, convex and weakly compact values of $\mathcal{U}^*$, it is straightforward to get that $\mathcal{T}u$ is nonempty, convex and weakly compact in $\mathcal{U}^*$ for each $u \in \mathcal{U}$.

The mapping $\mathcal{J}$ has a sequentially closed graph in $\mathcal{U}$ with weak- $\mathcal{U}$ topology. Taking in mind that it is also locally relatively weakly compact map in [34, Lemma 7.10], we have that $\mathcal{J}$ is also upper semicontinuous in this topology. Therefore, from $H(J)$ it follows that $\mathcal{T}$ is upper semicontinuous from every finite-dimensional subspace of $\mathcal{U}$ into $\mathcal{U}^*$ with weak topology.

Now let $y, y_n \in D(\mathcal{L})$ such that

$$\begin{aligned} y_n &\rightarrow y \text{ weakly in } \mathcal{U}, \\ \mathcal{L}(y_n) &\rightarrow \mathcal{L}(y) \text{ weakly in } \mathcal{U}^*, \\ y_n^* &\in \mathcal{B}y_n + \mathcal{J}(y_n), \\ y_n^* &\rightarrow y^* \text{ weakly in } \mathcal{U}. \end{aligned}$$

and we assume that $\limsup_n \langle y_n^*, y_n \rangle \leq \langle y^*, y \rangle$. Then,

$$y_n^* = u_n^* + w_n^* \text{ with } u_n^* = \mathcal{B}y_n, w_n^* \in \mathcal{J}(y_n).$$

Since the embedding $\mathcal{U} \subset \mathcal{H}$ is compact, then $y_n \rightarrow y$ in $\mathcal{U}$. Moreover, from the boundedness of the map $\mathcal{J}$, we pass (up to a subsequence, if necessary) to assume that there exists $w^* \in \mathcal{U}^*$ such that

$$w_n^* \rightarrow w^* \text{ weakly in } \mathcal{H}.$$

Since $\mathcal{J}$ has the sequentially closed graph, there is $w^* \in \mathcal{J}(y)$ such that

$$\begin{aligned} \limsup_n \langle u_n^*, y_n \rangle + \langle w^*, y \rangle &= \limsup_n \langle u_n^*, y_n \rangle + \lim_n \langle w_n^*, y_n \rangle \\ &= \limsup_n \langle u_n^* + w_n^*, y_n \rangle \leq \langle y^*, y \rangle. \end{aligned}$$

Then,

$$\limsup_n \langle u_n^*, y_n \rangle \leq \langle u^*, y \rangle,$$

here $u^* = y^* - w^*$. Furthermore, we have

$$u_n^* = y_n^* - w_n^* \rightarrow y^* - w^* = u^* \text{ weakly in } \mathcal{U}^*.$$

By hypothesis $H(B)$, $\mathcal{B}$ is $\mathcal{L}$ -generalized pseudomonotone, $u^* = \mathcal{B}y$ and one obtains that

$$\begin{aligned} \langle y_n^*, y_n \rangle &= \langle w_n^*, y_n \rangle + \langle u_n^*, y_n \rangle \\ &\rightarrow \langle w^*, y \rangle + \langle u^*, y \rangle = \langle y^*, y \rangle. \end{aligned}$$

So, $\mathcal{T}$ is a generalized pseudomonotone operator with respect to $D(\mathcal{L})$.

Moreover, $\mathcal{L}$ is linear and maximal monotone [35, Proposition 32.10]. Then based on Lemma 2.2, there exists $u \in \mathcal{W}$ such that:

$$\mathcal{L}u + \mathcal{T}u \ni 0.$$

Step 2: The uniqueness of solutions.

Now we shall verify that the system (3.1)–(3.2) possesses a unique solution. Given $x(\cdot) \in E$, let $u_i (i = 1, 2)$ be solutions of the problem (3.1)–(3.2) with $u_i(0) = u_0$. Therefore, we deduce that

$$u_i'(t) + B(t, u_i(t)) + M^* \xi_i(t) = g(t, x(t), u_i(t)),$$

where $\xi_i(t) \in \partial J(t, x(t), Mu_i(t))$. For $t \in [0, b]$, we subtract the above two identities, then multiply the result by $u_1(t) - u_2(t)$ to have

$$\begin{aligned} \frac{1}{2} |u_1(t) - u_2(t)|_H^2 &+ \int_0^t \langle B(s, u_1(s)) - B(s, u_2(s)), u_1(s) - u_2(s) \rangle ds \\ &+ \int_0^t \langle M^* \xi_1(s) - M^* \xi_2(s), u_1(s) - u_2(s) \rangle ds \\ &= \int_0^t \langle g(s, x(s), u_1(s)) - g(s, x(s), u_2(s)), u_1(s) - u_2(s) \rangle ds. \end{aligned}$$

Based on the monotonicity of the operator B w.r.t second variable and the assumption $H(g)$, we obtain that

$$\frac{1}{2} |u_1(t) - u_2(t)|_H^2 - \beta_J \int_0^t \|Mu_1(s) - Mu_2(s)\|_E^2 ds \leq \alpha_g \int_0^t |u_1(s) - u_2(s)|_H^2 ds,$$

which implies that

$$|u_1(t) - u_2(t)|_H^2 \leq 2(\beta_J \|M\|^2 + \alpha_g) \int_0^t |u_1(s) - u_2(s)|_H^2 ds.$$

It follows from Gronwall inequality that $u_1(t) = u_2(t)$ on $[0, b]$.

Step 3: The continuous dependence on initial values u_0 and the given function x .

We take $u_i(\cdot), i = 1, 2$ are the unique solutions to system (3.1)–(3.2) corresponding to $x_i(\cdot) \in C([0, b]; X), i = 1, 2$, that is,

$$\begin{aligned} u_1'(t) + B(t, u_1(t)) + M^* \xi_1(t) &= g(t, x_1(t), u_1(t)), \xi_1(t) \in \partial J(t, x_1(t), Mu_1(t)), \\ u_2'(t) + B(t, u_2(t)) + M^* \xi_2(t) &= g(t, x_2(t), u_2(t)), \xi_2(t) \in \partial J(t, x_2(t), Mu_2(t)). \end{aligned}$$

Let us multiply the above two equations by $u_1(t) - u_2(t)$ to have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_1(t) - u_2(t)|_H^2 + \frac{\beta_B}{\ell^2} |u_1(t) - u_2(t)|_H^2 \\ - \beta_J \|M\|^2 |u_1(t) - u_2(t)|_H^2 - \beta_J \|M\| \|x_1(t) - x_2(t)\|_X |u_1(t) - u_2(t)|_H \\ \leq \langle u_1'(t) - u_2'(t), u_1(t) - u_2(t) \rangle + \langle B(t, u_1(t)) - B(t, u_2(t)), u_1(t) - u_2(t) \rangle \\ + \langle M^* \xi_1(t) - M^* \xi_2(t), u_1(t) - u_2(t) \rangle \\ = \langle g(t, x_1(t), u_1(t)) - g(t, x_2(t), u_2(t)), u_1(t) - u_2(t) \rangle \\ \leq \alpha_g |u_1(t) - u_2(t)|_H^2 + \alpha_g \|x_1(t) - x_2(t)\|_X |u_1(t) - u_2(t)|_H, \end{aligned}$$

which implies that

$$\frac{1}{2} \frac{d}{dt} |u_1(t) - u_2(t)|_H^2 \leq -\beta |u_1(t) - u_2(t)|_H^2 + \rho \|x_1(t) - x_2(t)\|_X |u_1(t) - u_2(t)|_H,$$

where

$$\begin{aligned} \beta &= \frac{\beta_B}{\ell^2} - (\beta_J \|M\|^2 + \alpha_g) > 0 \\ \rho &= \|M\| \beta_J + \alpha_g > 0. \end{aligned}$$

Thus, it follows from the latter inequality that

$$\begin{aligned} \frac{d}{dt} (e^{2\beta t} |u_1(t) - u_2(t)|_H^2) &= 2\beta e^{2\beta t} |u_1(t) - u_2(t)|_H^2 + e^{2\beta t} \frac{d}{dt} |u_1(t) - u_2(t)|_H^2 \\ &\leq 2\rho e^{2\beta t} |u_1(t) - u_2(t)|_H \|x_1(t) - x_2(t)\|_X, \end{aligned}$$

By integrating this inequality from 0 to t , we yield

$$\begin{aligned} \frac{1}{2} e^{2\beta t} |u_1(t) - u_2(t)|_H^2 &\leq \frac{1}{2} |u_1(0) - u_2(0)|_H^2 \\ &+ \rho \int_0^t e^{2\beta s} \|x_1(s) - x_2(s)\|_X |u_1(s) - u_2(s)|_H ds. \end{aligned}$$

This allows us to infer that

$$\begin{aligned} (e^{\beta t} |u_1(t) - u_2(t)|_H)^2 &\leq |u_1(0) - u_2(0)|_H^2 \\ &+ \rho \int_0^t e^{\beta s} \|x_1(s) - x_2(s)\|_X |u_1(s) - u_2(s)|_H ds. \end{aligned}$$

This inequality together with [36, Lemma A.5, P. 157] leads that

$$e^{\beta t} |u_1(t) - u_2(t)|_H \leq |u_1(0) - u_2(0)|_H + \rho \int_0^t e^{\beta s} \|x_1(s) - x_2(s)\|_X ds,$$

which deduces that the continuous dependent estimate is true. The proof is complete. $\square$

Now it turns into the periodic problems for the hemivariational inequality as follows: Let $x \in C([0, b]; X)$ be given satisfying $x(0) = x(b)$, we find the solution $u(\cdot)$ such that

$$\begin{aligned} \langle u'(t), v \rangle + \langle B(t, u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \\ \geq \langle g(t, x(t), u(t)), v \rangle, \quad \forall v \in U, \text{ for a.e. } t \in [0, b], \end{aligned} \quad (3.4)$$

$$u(0) = u(b). \quad (3.5)$$

The following facts hold in our consideration.

(H_b) Assumptions H(J), H(B), H(M), H(g) and H(0) satisfy. In addition, the maps B, J, g are periodic in time with period b . That means:

$$\begin{aligned} B(0, u) &= B(b, u), \quad \forall u \in U; \\ J(0, x, u) &= J(b, x, u), \quad \forall x \in X, u \in U; \\ g(0, x, u) &= g(b, x, u), \quad \forall x \in X. \end{aligned}$$

Then we obtain the following results on the existence of a periodic solution to (3.4)–(3.5).

Theorem 3.2. *Let H(B), H(J), H(M), H(g), H(0) and (H_b) hold. Then the problem (3.4)–(3.5) has a unique solution. Moreover, we have the following estimates:*

$$|u_1(0) - u_2(0)|_H \leq \frac{\rho}{1 - e^{-\beta b}} \int_0^b e^{-\beta(b-s)} \|x_1(s) - x_2(s)\|_X ds, \quad (3.6)$$

where $u_1(\cdot), u_2(\cdot)$ are solutions of (3.4)–(3.5) corresponding to the given input functions $x_1(\cdot), x_2(\cdot)$. Furthermore, we have

$$\begin{aligned} |u_1(t) - u_2(t)|_H &\leq \frac{e^{-\beta t} \rho}{1 - e^{-\beta b}} \int_0^b e^{-\beta(b-s)} \|x_1(s) - x_2(s)\|_X ds \\ &\quad + \rho \int_0^t e^{-\beta(t-s)} \|x_1(s) - x_2(s)\|_X ds, \end{aligned} \quad (3.7)$$

for all $t \in [0, b]$.

Proof. Similarly as the proof of Lemma 3.1, we set

$$\begin{aligned} \mathcal{L}_1 : D(\mathcal{L}_1) \subset \mathcal{U} &\rightarrow \mathcal{U}^*, \\ \mathcal{L}_1 v &= v', \\ D(\mathcal{L}_1) &= \{v \in \mathcal{W} : v(0) = v(b)\}. \end{aligned}$$

It follows from Proposition 32.10 of Zeidler [35] that $\mathcal{L}_1$ is a densely defined maximal monotone function. By surjectivity lemma, we can obtain the existence of solutions to (3.4)–(3.5). Let $u_1(\cdot), u_2(\cdot)$ be the periodic solutions to (3.4)–(3.5) with respect to input functions $x_1(\cdot), x_2(\cdot) \in C([0, b]; X)$. Then it follows from the estimate (3.3) and the periodic conditions $u_1(0) = u_1(b), u_2(0) = u_2(b)$ that

$$|u_1(0) - u_2(0)|_H \leq e^{-\beta b} |u_1(0) - u_2(0)|_H + \rho \int_0^b e^{-\beta(b-s)} \|x_1(s) - x_2(s)\|_X ds,$$

which implies (3.6). A straightforward argument by using (3.3) again implies that (3.7) holds. The proof is complete. $\square$

4. Mild solutions to initial value problems for DHVIs

In this section we consider the following problem:

$$x'(t) = Ax(t) + (Rx)(t) + F(t, x(t), u(t)), \quad \text{for a.e. } t > 0, \quad (4.1)$$

$$\langle u'(t) + B(t, x(t), u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \geq \langle g(t, x(t), u(t)), v \rangle, \quad (4.2)$$

for all $v \in U$, for a.e. $t > 0$,

$$x(0) = x_0, \quad u(0) = u_0, \quad (4.3)$$

for given $(x_0, u_0) \in X \times U$. We add the condition for the linear operator A as follows:

H(A) A is a generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ on X .

H(F) $F : [0, b] \times X \times H \rightarrow X$ satisfies that

- (1) $F(\cdot, x, u)$ is measurable on $(0, b)$;
- (2) There are the nonnegative functions $m_F(\cdot), n_F(\cdot) \in L^1(0, b)$ such that

$$\|F(t, x, u) - F(t, y, v)\|_X \leq m_F(t)\|x - y\|_X + n_F(t)\|u - v\|_H,$$

for all $t \in [0, b], x, y \in X$ and $u, v \in U$.

H(R) $R : L^2(0, b; X) \rightarrow L^2(0, b; X)$ is a history-dependent operator, that is, there is $c_R > 0$ such that

$$\|(Rx)(t) - (Ry)(t)\|_X \leq c_R \int_0^t \|x(s) - y(s)\|_X ds,$$

for all $x, y \in L^2(0, b; X)$, a.e. $t \in (0, b)$.

Definition 4.1. A pair of functions $(x, u) \in C([0, b]; X) \times L^2(0, b; U)$ is called a mild solution to DHVI (4.1)–(4.3) if and only if the following facts hold:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)[(Rx)(s) + F(s, x(s), u(s))]ds \quad (4.4)$$

$$\langle u'(t) + B(t, x(t), u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \geq \langle g(t, x(t), u(t)), v \rangle, \quad (4.5)$$

for all $v \in U$, for a.e. $t > 0$,

$$u(0) = u_0, \quad (4.6)$$

Theorem 4.1. Suppose that H(A), H(B), H(F), H(R), H(J), H(M), H(g) and H(0) hold. Then there exists at least one mild solution for the problem (4.1)–(4.3). Moreover, this solution is unique.

Proof. Initially, we consider the system:

$$\langle u'(t) + B(t, x(t), u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \geq \langle g(t, x(t), u(t)), v \rangle, \quad (4.7)$$

for all $v \in U$, for a.e. $t > 0$,

$$u(0) = u_0, \quad (4.8)$$

Based on Lemma 3.1, the latter system has a unique solution $u(\cdot) \in \mathcal{U}$ and we have the estimate (3.3). We denote by $SOL(x, u_0)$ the solution of (4.7)–(4.8) with respect to the initial datum u_0 and the given function $x(\cdot)$. Then we can rewrite (4.4)–(4.6) by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)[(Rx)(s) + F(s, x(s), SOL(x, u_0)(s))]ds, \quad (4.9)$$

$$u(t) = SOL(x, u_0)(t). \quad (4.10)$$

Let us be fixed x_0, u_0 , we consider the solution map by

$$\begin{aligned} \mathcal{G} : C([0, b]; X) &\rightarrow C([0, b]; X), \\ \mathcal{G}(x)(t) &= S(t)x_0 + \int_0^t S(t-s)[(Rx)(s) + F(s, x(s), SOL(x, u_0)(s))]ds. \end{aligned}$$

Then we have the following estimates:

$$\begin{aligned} \|F(s, x(s), SOL(x, u_0)(s))\|_X &\leq m_F(s)\|x(s)\|_X + n_F(s)|SOL(x, u_0)(s)|_H \\ &\quad + \|F(0, 0, 0)\|_X \\ &\leq m_F(s)\|x(s)\|_X + n_F(s)(e^{-\beta s}|u_0|_H + \rho \int_0^s \|x(\tau)\|_X d\tau) \\ &\quad + \|F(0, 0, 0)\|_X \\ &\leq m_F(s)\|x(s)\|_X + n_F(s)\rho \int_0^s \|x(\tau)\|_X d\tau \\ &\quad + n_F(s)|u_0|_H + \|F(0, 0, 0)\|_X, \end{aligned}$$

thanks to $H(F)$. Moreover, it follows from $H(R)$ that

$$\|(Rx)(s)\|_X \leq c_R \int_0^s \|x(\tau)\|_X d\tau + \|(R0)(s)\|_X.$$

Combining the above estimates, we get:

$$\begin{aligned} \|\mathcal{G}(x)(t)\|_X &\leq P\|x_0\|_X + P \int_0^t \left(c_R \int_0^s \|x(\tau)\|_X d\tau + \|(R0)(s)\|_X \right) ds \\ &\quad + P \int_0^t \left(m_F(s)\|x(s)\|_X + n_F(s)\rho \int_0^s \|x(\tau)\|_X d\tau \right) ds \\ &\quad + P \int_0^t (n_F(s)|u_0|_H + \|F(0, 0, 0)\|_X) ds \\ &\leq P\|x_0\|_X + Pbc_R \int_0^t \|x(s)\|_X ds + P \int_0^t \|(R0)(s)\|_X ds \\ &\quad + P \int_0^t m_F(s)\|x(s)\|_X ds + P\rho \int_0^t n_F(s)ds \int_0^s \|x(\tau)\|_X d\tau \\ &\quad + P \int_0^t (n_F(s)|u_0|_H + \|F(0, 0, 0)\|_X) ds \\ &\leq \int_0^t m_{1\mathcal{G}}(s)\|x(s)\|_X ds + m_{2\mathcal{G}}(t), \end{aligned}$$

where

$$\begin{aligned} P &= \sup_{t \in [0, b]} \|S(t)\|_{\mathcal{L}(X)}, \\ m_{1\mathcal{G}}(s) &= P(bc_R + m_F(s) + \rho\|n_F\|_{L^1(0, b)}), \\ m_{2\mathcal{G}}(t) &= P(\|x_0\|_X + \|(R0)\|_{L^1(0, t)} + |u_0|_H\|\eta_F\|_{L^1(0, t)}) + Pt\|F(0, 0, 0)\|_X. \end{aligned}$$

Denote

$$\mathcal{D}_0 = \{x \in C([0, b]; X) : \|x(t)\|_X \leq \varpi(t)\},$$

where $\varpi(\cdot)$ is the solution of the following integral inequality:

$$\varpi(t) \leq \int_0^t m_{1\mathcal{G}}(s)\varpi(s)ds + m_{2\mathcal{G}}(t).$$

Then it is obvious that $\mathcal{D}_0$ is a closed, convex subset of $C([0, b]; X)$ and

$$\mathcal{G}(\mathcal{D}_0) \subset \mathcal{D}_0.$$

Set

$$\mathcal{D}_{k+1} = \overline{\text{co}}\mathcal{G}(\mathcal{D}_k), k = 0, 1, 2, \dots$$

here, the notation $\overline{\text{co}}$ refers to the closure of the convex hull of a subset in $C([0, b]; X)$. It is easy to see that $\mathcal{D}_k$ is closed and convex. Furthermore,

$$\mathcal{D}_{k+1} \subset \mathcal{D}_k \text{ for all } k \in \mathbb{N}.$$

Let $\mathcal{D} = \bigcap_{k=0}^{\infty} \mathcal{D}_k$, we have that $\mathcal{D} \neq \emptyset$. Indeed, we will begin to prove that $\mu_k(t) := \chi_X(\mathcal{D}_k(t)) \rightarrow 0$ as $k \rightarrow \infty$, here χ_X stands for the Hausdorff measure of noncompactness on X .

If $\{S(t)\}$ is compact, then $\mu_k(t) = 0, \forall t \geq 0$. On the other side, we use the estimate regarding to MNC as follows:

$$\begin{aligned} \mu_{k+1}(t) &\leq \chi_X \left(\int_0^t S(t-s)(R\mathcal{D}_k)(s)ds \right) \\ &\quad + \chi_X \left(\int_0^t S(t-s)F(s, \mathcal{D}_k(s), SOL(\mathcal{D}_k, u_0)(s))ds \right) \\ &\leq 4P \int_0^t \chi_X(R\mathcal{D}_k)(s)ds + 4P \int_0^t \chi_X(F(s, \mathcal{D}_k(s), SOL(\mathcal{D}_k, u_0)(s)))ds, \end{aligned}$$

thanks to [Proposition 2.4](#). Let $x, y \in C([0, b]; X)$, then,

$$\begin{aligned} &\| F(s, x(s), SOL(x, u_0)(s)) - F(s, y(s), SOL(y, u_0)(s)) \|_X \leq m_F(s)\|x(s) - y(s)\|_X \\ &\quad + n_F(s)|SOL(x, u_0)(s) - SOL(y, u_0)(s)|_H \\ &\leq m_F(s)\|x(s) - y(s)\|_X + \rho n_F(s) \int_0^s e^{-\beta(s-\tau)}\|x(\tau) - y(\tau)\|_X d\tau \\ &\leq m_F(s)\|x(s) - y(s)\|_X + \rho n_F(s) \int_0^s \|x(\tau) - y(\tau)\|_X d\tau. \end{aligned}$$

Moreover,

$$\chi_X(R\mathcal{D}_k)(s) \leq 4c_R \int_0^s \chi_X(\mathcal{D}_k(\tau))d\tau.$$

Therefore,

$$\chi_X(F(s, \mathcal{D}_k(s), SOL(\mathcal{D}_k, u_0)(s))) \leq m_F(s)\chi_X(\mathcal{D}_k(s)) + 4\rho n_F(s) \int_0^s \chi_X(\mathcal{D}_k(\tau))d\tau,$$

which implies that

$$\begin{aligned} &\int_0^t \chi_X(R\mathcal{D}_k)(s)ds + \int_0^t \chi_X(F(s, \mathcal{D}_k(s), SOL(\mathcal{D}_k, u_0)(s)))ds \\ &\leq \int_0^t (m_F(s) + 4c_R b)\chi_X(\mathcal{D}_k(s))ds + 4\rho \int_0^t n_F(s) \int_0^s \chi_X(\mathcal{D}_k(\tau))d\tau ds \\ &\leq \int_0^t \left(m_F(s) + 4c_R b + 4\rho \|n_F\|_{L^1(0, b)} \right) \chi_X(\mathcal{D}_k(s))ds, \end{aligned}$$

due to Proposition 2.4 again. Thus,

$$\mu_{k+1}(t) \leq 4P \int_0^t \left(m_F(s) + 4c_R b + 4\rho \|n_F\|_{L^1(0,b)} \right) \chi_X(\mathcal{D}_k(s)) ds.$$

It is equivalently

$$\mu_{k+1}(t) \leq 4P \int_0^t \left(m_F(s) + 4c_R b + 4\rho \|n_F\|_{L^1(0,b)} \right) \mu_k(s) ds.$$

Putting $\mu_\infty(t) = \lim_{k \rightarrow \infty} \mu_k(t)$ and passing to the limit, we obtain

$$\mu_\infty(t) \leq 4P \int_0^t \left(m_F(s) + 4c_R b + 4\rho \|n_F\|_{L^1(0,b)} \right) \mu_\infty(s) ds.$$

It can be deduced from the Gronwall inequality that $\mu_\infty(t) = 0$ for all $t \in (0, b)$. Now, we prove that $\mathcal{D} \neq \emptyset$. Let $x_k \in \mathcal{D}_k$, $k \geq 1$ as follows

$$\begin{aligned} \hat{x}_0 &= 0 \in \mathcal{D}_0, \\ x_1 &= \mathcal{G}(\hat{x}_0), \\ &\dots \\ x_{k+1} &= \mathcal{G}(x_k). \end{aligned}$$

Then, we have

$$\chi_X(\{x_k(t) : k \in \mathbb{N}\}) = \chi_X(\{x_k(t) : k \in \mathbb{N}, k \geq N\}), \forall N \in \mathbb{N},$$

$$\chi_X(\{x_k(t) : k \in \mathbb{N}, k \geq N\}) \leq \chi_X(\mathcal{D}_N(t)) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Thus,

$$\chi_X(\{x_k(t) : k \in \mathbb{N}\}) = 0.$$

On the other hand, $\{x_k\}$ is integrably bounded in view of the integrable boundedness of $\mathcal{D}_0$. Then, based on [25, Proposition 3.5], the set

$$\left\{ \int_0^t S(t-s)(Rx_k)(s)ds + \int_0^t S(t-s)F(s, x_k(s), SOL(x_k, u_0)(s))ds : k \geq 1 \right\}$$

is relatively compact in $C([0, b]; X)$. We deduce that

$$\begin{aligned} \mathcal{G}(\{x_k : k \in \mathbb{N}\}) &= S(t)x_0 + \left\{ \int_0^t S(t-s)(Rx_k)(s)ds \right. \\ &\quad \left. + \int_0^t S(t-s)F(s, x_k(s), SOL(x_k, u_0)(s))ds : k \geq 1 \right\} \end{aligned}$$

is also relatively compact. Hence, by taking a subsequence of $\{x_k\}_{k \geq 1}$, without generality, assume that $x_k \rightarrow x^*$ as $k \rightarrow \infty$ in $C([0, b]; X)$. Thus $x^* \in \mathcal{D}$, so $\mathcal{D} \neq \emptyset$.

It is easily seen that $\mathcal{D}$ is a closed convex subset of $C([0, b]; X)$ and the relation

$$\mathcal{G}(\mathcal{D}) \subset \mathcal{D}$$

holds. Moreover, by the aid of the growth property of F then for each $k \geq 1$, $F(s, \mathcal{D}(s), SOL(\mathcal{D}, u_0)(s))$ is integrably bounded. Thus, $\mathcal{D}$ is an integrably bounded set.

As $\mu_\infty(t) = 0$ for all $t \in [0, b]$, we obtain that $\mathcal{D}(t)$ is relatively compact for all $t \in [0, b]$. By [25, Proposition 3.5], $\left\{ \int_0^\cdot S(\cdot-s)F(s, \mathcal{D}(s), SOL(\mathcal{D}, u_0)(s))ds \right\}$ is a relatively compact subset in $C([0, b]; X)$. Then $\mathcal{G}(\mathcal{D})$ is also relatively compact in $C([0, b]; X)$.

Denote

$$\mathcal{O} = \overline{\text{co}}\mathcal{G}(\mathcal{D}).$$

It can be easily seen that $\mathcal{O}$ is a nonempty compact convex subset of $C([0, T]; X)$ and $\mathcal{G}(\mathcal{O}) \subset \mathcal{O}$ because $\mathcal{G}(\mathcal{O}) = \mathcal{G}(\overline{\text{co}}\mathcal{G}(\mathcal{D})) \subset \mathcal{G}(\mathcal{D}) \subset \overline{\text{co}}\mathcal{G}(\mathcal{D}) = \mathcal{O}$.

We now consider the mapping $\mathcal{G} : \mathcal{O} \rightarrow \mathcal{O}$. In what follows, we prove $\mathcal{G}$ is a continuous map. Let $x_n \in \mathcal{O}$ with $x_n \rightarrow x^*$ and $y_n = \mathcal{G}(x_n)$ with $y_n \rightarrow y^*$. Then

$$y_n(t) = S(t)x_0 + \int_0^t S(t-s)(Rx_n)(s)ds + \int_0^t S(t-s)F(s, x_n(s), \text{SOL}(x_n, u_0)(s))ds.$$

By the continuity of F and the condition $H(R)$ we can pass to the limit to get that

$$x^*(t) = S(t)x_0 + \int_0^t S(t-s)(Rx^*)(s)ds + \int_0^t S(t-s)F(s, x^*(s), \text{SOL}(x^*, u_0)(s))ds.$$

Taking in mind Lemma 2.3, the preceding Lemma 2.3, one obtains that $\mathcal{G}$ admits a fixed point $\mathbf{x}$.

Let $\mathbf{u}(\cdot) = \text{SOL}(\mathbf{x}, u_0)$, we assure that $(\mathbf{x}, \mathbf{u})$ is a mild solution for (4.1)–(4.3). The uniqueness result is followed from the Lipschitz type condition of the perturbation F . $\square$

From the compact embedding $U \subset\subset H$, if $u_0 \in H$ then there exists a sequence $\{u_{n0}\} \subset U$ such that $u_{n0} \rightarrow u_0$ in H . By Lemma 3.1, for each $x \in C([0, b]; X)$, there is a sequence $\{u_n\}$ such that u_n is the solution of (3.1)–(3.2) with initial condition u_{n0} and $\{u_n\}$ converges to $u \in C([0, b]; H)$. Moreover, u is unique for all possible choices of sample sequence u_{n0} . In this case, for $(x_0, u_0) \in X \times H$, we can prove that the system (4.1)–(4.3) has a unique mild solution and the estimate (3.3) is also valid.

5. Mild solutions to periodic problems for DHVIs

In this section we shall investigate the existence of mild solution to the problem (1.1)–(1.3). We replace $H(A)$ by $H(A^*)$ and we add the condition H_b^* as follows:

$H(A^*)$ A satisfies $H(A)$ and the semigroup $S(t)$ is exponentially stable with exponent γ :

$$\|S(t)\|_{\mathcal{L}(X)} \leq e^{-\gamma t}, \forall t \geq 0.$$

(H_b^*) The hypothesis H_b satisfies and $F(0, x, u) = F(b, x, u)$ for every $x \in X, u \in H$.

For given $x_0, y_0 \in X$ and $u_0, v_0 \in H$, we consider the problem finding the mild solutions $(x(\cdot), u(\cdot))$ and $(y(\cdot), v(\cdot))$ of the following systems:

$$x'(t) = Ax(t) + (Rx)(t) + F(t, x(t), u(x)(t)), \quad (5.1)$$

$$\langle u'(t) + B(t, u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \geq \langle g(t, x(t), u(t)), v \rangle, \forall v \in U, \quad (5.2)$$

$$x(0) = x_0, \quad u(0) = u_0, \quad (5.3)$$

and

$$x'(t) = Ax(t) + (Rx)(t) + F(t, x(t), u(x)(t)), \quad (5.4)$$

$$\langle u'(t) + B(t, u(t)), v \rangle + J^0(t, x(t), Mu(t); Mv) \geq \langle g(t, x(t), u(t)), v \rangle, \forall v \in U, \quad (5.5)$$

$$x(0) = y_0, \quad u(0) = v_0. \quad (5.6)$$

For $t \in [0, b]$, we have the following estimates:

$$\begin{aligned} \|x(t) - y(t)\|_X &\leq \|S(t)(x_0 - y_0)\|_X + \int_0^t \|S(t-s)[(Rx)(s) - (Ry)(s)]\|_X ds \\ &\quad + \int_0^t \|S(t-s)[F(s, x(s), u(s)) - F(s, y(s), v(s))]\|_X ds \\ &\leq e^{-\gamma t} \|x_0 - y_0\|_X + \int_0^t e^{-\gamma(t-s)} \|(Rx)(s) - (Ry)(s)\|_X ds \\ &\quad + \int_0^t e^{-\gamma(t-s)} \|F(s, x(s), u(s)) - F(s, y(s), v(s))\|_X ds \\ &\leq e^{-\gamma t} \|x_0 - y_0\|_X + \int_0^t e^{-\gamma(t-s)} c_R \int_0^s \|x(\tau) - y(\tau)\|_X d\tau ds \\ &\quad + \int_0^t e^{-\gamma(t-s)} [m_F^* \|x(s) - y(s)\|_X + n_F^* |u(s) - v(s)|_H] ds \\ &\leq e^{-\gamma t} \|x_0 - y_0\|_X + (m_F^* + bc_R) \int_0^t \|x(s) - y(s)\|_X ds \\ &\quad + n_F^* \int_0^t |u(s) - v(s)|_H ds, \end{aligned}$$

where $m_F^* = \sup_{[0,b]} m_F(t)$ and $n_F^* = \sup_{[0,b]} n_F(t)$. Moreover, one has

$$|u(t) - v(t)|_H \leq e^{-\beta t} |u_0 - v_0|_H + \rho \int_0^t e^{-\beta(t-s)} \|x(s) - y(s)\|_X ds.$$

Then combining the latter inequalities to obtain that

$$\begin{aligned} \|x(t) - y(t)\|_X + |u(t) - v(t)|_H &\leq e^{-\vartheta t} (\|x_0 - y_0\|_X + |u_0 - v_0|_H) \\ &\quad + \kappa \int_0^t (\|x(s) - y(s)\|_X + |u(s) - v(s)|_H) ds, \end{aligned}$$

where $\kappa := \max\{m_F^* + bc_R + \rho; n_F^*\}$ and $\vartheta := \max\{\gamma; \beta\}$. Inspired by Gronwall–Bellman inequality, we have

$$\begin{aligned} \|x(t) - y(t)\|_X + |u(t) - v(t)|_H &\leq e^{-\vartheta t} (\|x_0 - y_0\|_X + |u_0 - v_0|_H) \\ &\quad + \int_0^t e^{-\vartheta s} (\|x_0 - y_0\|_X + |u_0 - v_0|_H) \kappa e^{\kappa(t-s)} ds \\ &= e^{-\vartheta t} (\|x_0 - y_0\|_X + |u_0 - v_0|_H) \\ &\quad + \frac{\kappa e^{\kappa t}}{\vartheta + \kappa} (\|x_0 - y_0\|_X + |u_0 - v_0|_H) (1 - e^{-(\vartheta + \kappa)t}) \\ &= \left(\frac{\vartheta e^{-\vartheta t}}{\vartheta + \kappa} + \frac{\kappa e^{\kappa t}}{\vartheta + \kappa} \right) (\|x_0 - y_0\|_X + |u_0 - v_0|_H). \end{aligned}$$

Theorem 5.1. Assume that assumptions $H(A^*)$, $H(F)$, $H(R)$, $H(B)$, $H(F)$, $H(R)$, $H(J)$, $H(M)$, $H(g)$, $H(0)$ and (H_b^*) are fulfilled. Then the history-dependent DHVI (1.1)–(1.2) has a unique mild periodic solution with period b , provided that:

$$\varkappa := e^{-\vartheta b} + \frac{\kappa(\vartheta + \kappa e^{\kappa b} - \vartheta e^{-\vartheta b} - \kappa)}{\vartheta + \kappa} < 1. \quad (5.7)$$

Proof. Since the existence and uniqueness results stated in Theorem 4.1, we can put

$$\mathbb{G} : X \times H \rightarrow X \times H;$$

$$\mathbb{G}(x_0, u_0) = (x(b), u(b)),$$

where $(x(\cdot), u(\cdot))$ is the (unique) mild solution of the problem (1.1)–(1.3) with initial condition $x(0) = x_0$, $u(0) = u_0$.

We can prove that $\mathbb{G}$ is a contraction mapping in $X \times H$. In fact, we have

$$\begin{aligned} \|\mathbb{G}(x_0, u_0) - \mathbb{G}(y_0, v_0)\|_{X \times H} &\leq e^{-\vartheta b} (\|x_0 - y_0\|_X + |u_0 - v_0|_H) \\ &\quad + \kappa \int_0^b (\|x(s) - y(s)\|_X + |u(s) - v(s)|_H) ds \\ &\leq e^{-\vartheta b} (\|x_0 - y_0\|_X + |u_0 - v_0|_H) \\ &\quad + \kappa (\|x_0 - y_0\|_X + |u_0 - v_0|_H) \int_0^b \left(\frac{\vartheta e^{-\vartheta s}}{\vartheta + \kappa} + \frac{\kappa e^{\kappa s}}{\vartheta + \kappa} \right) ds \\ &\leq \varkappa (\|x_0 - y_0\|_X + |u_0 - v_0|_H). \end{aligned}$$

Taking into account $0 < \varkappa < 1$, then $\mathbb{G}$ is a contraction mapping. Then it follows from the Banach fixed-point theorem that $\mathbb{G}$ has a unique fixed point (x^*, u^*) , namely $\mathbb{G}(x^*, u^*) = (x^*, u^*)$. This fixed point is the initial value from which the mild b -periodic $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ solution of the problem (1.1)–(1.2) starts. The proof is complete. $\square$

6. Applications

In this section we apply our previous results to consider the periodic problems to a contact problem in solid mechanics under the impact of temperature, which lead to a history-dependent differential hemivariational inequality. Let Ω be a bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with C^2 -smooth boundary $\partial\Omega$. The boundary $\partial\Omega$ is composed of three subsets $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ with the mutually disjoint relatively open sets Γ_1 , Γ_2 , Γ_3 such that $meas(\Gamma_1) > 0$.

Here and below we are boldface letters for vectors and tensors. The notation $\boldsymbol{\nu}$ denotes the outward unit normal on Γ and $i, j = \overline{1, d}$. We also denote the partial derivative with respect to the corresponding component of the spatial variable $\mathbf{x}$ by the comma indicates, and sometimes omit the dependence of variable $\mathbf{x}$ for simplicity. Moreover, we denote the space of second order symmetric tensors on $\mathbb{R}^d$ by $\mathbb{S}^d$, and the zero element of $\mathbb{R}^d$ and $\mathbb{S}^d$ by $\mathbf{0}$. We introduce the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_j, \quad \|\mathbf{u}\| = (\mathbf{u} \cdot \mathbf{u})^{\frac{1}{2}} \quad \text{for all } \mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

Also, the deformation operator $\boldsymbol{\varepsilon}$ and divergence operator Div are formulated as follows

$$\begin{aligned} \varepsilon_{ij}(\mathbf{u}) &= \frac{1}{2}(u_{i,j} + u_{j,i}), \quad u_{i,j} = \frac{\partial}{\partial x_i} u_j, \quad u_{j,i} = \frac{\partial}{\partial x_j} u_i, \\ \boldsymbol{\varepsilon}(\mathbf{u}) &= (\varepsilon_{ij}(\mathbf{u})), \\ Div \boldsymbol{\sigma} &= (\sigma_{ij,j}). \end{aligned}$$

Furthermore, the indices ν and τ represent normal components and tangential parts of vectors and tensors. The normal and tangential components of:

- (i) a vector $\mathbf{u}$ on Γ are formulated by $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$ and $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$;
- (ii) a tensor $\boldsymbol{\sigma}$ on Γ are formulated by $\sigma_\nu = \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

We consider the contact problem with normal compliance driven by a history-dependent dynamical system stated in the following.

Find a displacement field $\mathbf{u} : [0, b] \times \Omega \rightarrow \mathbb{R}^d$, a stressfield $\boldsymbol{\sigma} : [0, b] \times \Omega \rightarrow \mathbb{S}^d$ and a temperature $\theta : [0, b] \times \Omega \rightarrow \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{F}(t, \boldsymbol{\varepsilon}(\mathbf{u}(t))) \text{ in } \mathbf{x} \in \Omega, t \in (0, b), \quad (6.1)$$

$$\mathbf{u}'(t) + \text{Div } \boldsymbol{\sigma}(t) = \mathbf{f}_0(t, \theta(t)) \text{ in } \mathbf{x} \in \Omega, t \in (0, b), \quad (6.2)$$

$$\mathbf{u}(t) = \mathbf{0} \text{ on } \Gamma_1, t > 0, \quad (6.3)$$

$$\boldsymbol{\sigma}(t) \boldsymbol{\nu} = \mathbf{f}_2(t) \text{ on } \Gamma_2, t \in (0, b), \quad (6.4)$$

$$-\sigma_\nu(t) \in \partial j_\nu(t, \theta(t), u_\nu(t)) \text{ on } \Gamma_3, t \in (0, b), \quad (6.5)$$

$$\boldsymbol{\sigma}_\tau(t) = \mathbf{0} \text{ on } \Gamma_3, t \in (0, b), \quad (6.6)$$

$$\theta'(t) - \Delta \theta(t) = f(t, \theta(t), \mathbf{u}(t)) + \chi \int_0^t \theta(s) ds; \text{ in } \mathbf{x} \in \Omega, t \in (0, b), \quad (6.7)$$

$$\theta(t) = 0 \text{ on } \partial \Omega, t \geq 0, \quad (6.8)$$

$$\theta(0) = \theta(b), \mathbf{u}(0) = \mathbf{u}(b). \quad (6.9)$$

We now present a short description of the equations. First, Eq. (6.1) is the viscoelastic constitutive law for elastic materials in which $\mathcal{F}$ represents the elasticity operator, $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor. Eq. (6.2) represents the motion equation in which $\mathbf{f}_0$ represents the density of the body forces being impacted on temperature θ . Conditions (6.3) and (6.4) represent the classical displacement-traction boundary conditions. They show that the body is fixed on Γ_1 and surface tractions of density $\mathbf{f}_2$ act on Γ_2 , respectively. The inclusion (6.5) represents the contact law with normal compliance under the influence of temperature θ , in which ∂j_ν denotes the Clarke subdifferential of a given function j_ν . It is to emphasize that this condition models the contact with a deformable foundation. Finally, the relations (6.7)–(6.9) represent the temperature θ as well as the periodic conditions of our system, where χ is a given positive number.

In order to investigate the above system, we establish the following hypotheses.

($\mathcal{H}_{\mathcal{F}}$) The elasticity operator $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ satisfying

- (i) there exists $L_{\mathcal{F}} > 0$ such that for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega, t > 0$,

$$\|\mathcal{F}(\mathbf{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, t, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|;$$

- (ii) there exists $m_{\mathcal{F}} > 0$ such that for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega, t > 0$,

$$[\mathcal{F}(\mathbf{x}, t, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, t, \boldsymbol{\varepsilon}_2)] : (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2;$$

- (iii) for all $t > 0, \boldsymbol{\varepsilon} \in \mathbb{S}^d$, $\mathcal{F}(\cdot, t, \boldsymbol{\varepsilon})$ is measurable in Ω and for a.e. $(\mathbf{x}, t) \in \Omega \times \mathbb{R}$, $\mathcal{F}(\mathbf{x}, t, \cdot)$ is continuous;
- (iv) $\mathcal{F}(\mathbf{x}, t, \mathbf{0}) = \mathbf{0}$ for a.e. $\mathbf{x} \in \Omega$.
- (v) $\mathcal{F}(\mathbf{x}, 0, \boldsymbol{\varepsilon}) = \mathcal{F}(\mathbf{x}, b, \boldsymbol{\varepsilon})$ for all $\mathbf{x} \in \Omega, \boldsymbol{\varepsilon} \in \mathbb{S}^d$.

($\mathcal{H}_f$) For the densities of body forces $\mathbf{f}_0$ and $\mathbf{f}_2$ which $\mathbf{f}_0$ depends on the temperature, $\mathbf{f}_0 : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^d$, we assume

- (i) $\mathbf{f}_0(\cdot, r, s) \in L^2(\Omega, \mathbb{R}^d)$ for every $r, s \in \mathbb{R}$;
- (ii) $\mathbf{f}_0(\mathbf{x}, \cdot, \cdot)$ is a continuous function in $\mathbb{R}^2$ for every $\mathbf{x} \in \Omega$. Moreover, $\mathbf{f}_0(\mathbf{x}, 0, r) = \mathbf{f}_0(\mathbf{x}, b, r)$ for all $\mathbf{x} \in \Omega, r \in \mathbb{R}$,

- (iii) $\|\mathbf{f}_0(\mathbf{x}, t, y) - \mathbf{f}_0(\mathbf{x}, t, z)\|_d \leq \alpha_1 |y - z|$ for all $\mathbf{x} \in \Omega, t > 0, y, z \in \mathbb{R}$ for some $\alpha_1 > 0$.
- (iv) $\mathbf{f}_2(t) \in L^2(\Gamma_2, \mathbb{R}^d), t > 0$ satisfies:

$$\|\mathbf{f}_2(t, y) - \mathbf{f}_2(t, z)\|_d \leq \alpha_2 |y - z|,$$

for all $t > 0, y, z \in \mathbb{R}$ for some $\alpha_2 > 0$.

($\mathcal{H}_{j_\nu}$) $j_\nu : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is such that

- (i) $j_\nu(\cdot, \cdot, q, r)$ is measurable on $\Gamma_3 \times (0, b)$ for all $q, r \in \mathbb{R}$,
- (ii) $j_\nu(\mathbf{x}, q, r, \cdot)$ is locally Lipschitz on $\mathbb{R}$, for each $\mathbf{x} \in \Omega, q, r \in \mathbb{R}$.
- (iii) $|\partial j_\nu(\mathbf{x}, t, q, r)| < \kappa_{j_\nu}(\mathbf{x}, t) + \kappa_1 |q| + \kappa_2 |r|$ for some function $\kappa_{j_\nu} \in L^2(\Gamma_3 \times (0, b); \mathbb{R}^+)$ and for constants $\kappa_1 \geq 0, \kappa_2 \geq 0$,
- (iv) the following estimate holds

$$(v_1 - v_2)(r_1 - r_2) \geq -\bar{\beta}_1 |r_1 - r_2|^2 - \bar{\beta}_2 |r_1 - r_2| |q_1 - q_2|,$$

for all $\mathbf{x} \in \Gamma_3$ and $t, q_i, r_i \in \mathbb{R}, i = 1, 2, v_i \in \partial j_\nu(\mathbf{x}, t, q_i, r_i), i = 1, 2$,

- (vi) $j_\nu(\mathbf{x}, 0, q, r) = j_\nu(\mathbf{x}, b, q, r)$ for all $\mathbf{x} \in \Omega, q, r \in \mathbb{R}$.

($\mathcal{H}_f^*$) The function $f : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying

- (i) $f(\mathbf{x}, \cdot, \cdot, \cdot)$ is continuous function on $\mathbb{R} \times \mathbb{R}^d$, for all $\mathbf{x} \in \Omega$,
- (ii) $f(\cdot, p, q, \hat{\mathbf{x}}) \in L^2(\Omega)$, for all $p, q \in \mathbb{R}, \hat{\mathbf{x}} \in \mathbb{R}^d$,
- (iii) there are nonnegative constants ν_1, ν_2 such that

$$|f(\mathbf{x}, p, q, \hat{\mathbf{x}}) - f(\mathbf{x}, p, r, \hat{\mathbf{y}})| \leq \nu_1 |q - r| + \nu_2 \|\hat{\mathbf{x}} - \hat{\mathbf{y}}\|,$$

for all $\mathbf{x}, \mathbf{y} \in \Omega, p, q, r \in \mathbb{R}, \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{R}^d$.

- (iv) $f(\mathbf{x}, 0, p, \hat{\mathbf{x}}) = f(\mathbf{x}, b, p, \hat{\mathbf{x}})$ for all $\mathbf{x} \in \Omega, p \in \mathbb{R}, \hat{\mathbf{x}} \in \mathbb{S}^d$.

The abstract spaces given by the followings:

$$\begin{aligned} X &= L^2(\Omega); \\ U &= \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^d) | \mathbf{u} = \mathbf{0} \text{ a.e. on } \Gamma_1\}; \\ H &= L^2(\Omega, \mathbb{R}^d); \\ E &= L^2(\Gamma_3, \mathbb{R}^d); \\ U^* &= H^{-1}(\Omega, \mathbb{R}^d). \end{aligned}$$

Since $\text{meas}(\Gamma_1) > 0$, it follows from the Korn and Poincare inequalities that U endowed with the inner product and the norm

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} \rangle &= \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) d\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in U, \\ \|\mathbf{u}\|_U &= \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{u}) d\mathbf{x}, \quad \forall \mathbf{u} \in U, \end{aligned}$$

is a Hilbert space. Moreover, H is a Hilbert equipped with the inner product:

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_H = \int_{\Omega} \boldsymbol{\sigma}(\mathbf{x}) : \boldsymbol{\tau}(\mathbf{x}) d\mathbf{x}, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in H.$$

We introduce the following operator:

$$A = \Delta_D;$$

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

It is known that from [37], the semigroup $S(t) = e^{tA}$ generated by A is compact and exponentially stable, namely,

$$\|S(t)\|_{\mathcal{L}(X)} \leq e^{-\lambda_1 t}, \quad \forall t \geq 0,$$

where λ_1 is the first eigenvalue of the Laplacian operator in the Dirichlet boundary condition. In this case, $\gamma = \lambda_1$ and the condition $H(A^*)$ holds.

Let $\mathbf{v} \in U$, using the Green formula, then we split the surface integral on three integrals on Γ_1, Γ_2 and Γ_3 and we use the identities (6.2), (6.4) to deduce that

$$\begin{aligned} \int_{\Omega} \mathbf{u}' \cdot \mathbf{v} d\mathbf{x} + \int_{\Omega} \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) d\mathbf{x} &= \int_{\Omega} \mathbf{f}_0(t, \theta(t)) \cdot (\mathbf{v} - \mathbf{u}) d\mathbf{x} + \int_{\Gamma_1} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}) d\Gamma \\ &+ \int_{\Gamma_2} \mathbf{f}_2(t) \cdot (\mathbf{v} - \mathbf{u}) d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}) d\Gamma. \end{aligned}$$

In view of the following identities

$$\begin{aligned} \mathbf{v} - \mathbf{u} &= \mathbf{0} \quad \text{a.e. on } \Gamma_1, \\ \boldsymbol{\sigma} \boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}) &= \boldsymbol{\sigma}_{\tau} \cdot (\mathbf{v}_{\tau} - \mathbf{u}_{\tau}) + \sigma_{\nu}(v_{\nu} - u_{\nu}) \quad \text{a.e. on } \Gamma_3, \end{aligned}$$

we deduce that

$$\begin{aligned} \int_{\Omega} \mathbf{u}' \cdot \mathbf{v} d\mathbf{x} + \int_{\Omega} \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) d\mathbf{x} &= \int_{\Omega} \mathbf{f}_0(t, \theta(t)) \cdot (\mathbf{v} - \mathbf{u}) \\ &+ \int_{\Gamma_2} \mathbf{f}_2(t) \cdot (\mathbf{v} - \mathbf{u}) d\Gamma + \int_{\Gamma_3} \sigma_{\nu}(v_{\nu} - u_{\nu}) d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}_{\tau} \cdot (\mathbf{v}_{\tau} - \mathbf{u}_{\tau}) d\Gamma. \end{aligned} \quad (6.10)$$

On the other hand, it follows from the contact boundary condition (6.5) and the definition of the Clark subdifferential that

$$\sigma_{\nu}(v_{\nu} - u_{\nu}) \leq j_{\nu}^0(t, \theta(t), u_{\nu}(t); v_{\nu} - u_{\nu}) \quad \text{a.e. on } \Gamma_3,$$

therefore,

$$\int_{\Gamma_3} \sigma_{\nu}(v_{\nu} - u_{\nu}) d\Gamma \geq - \int_{\Gamma_3} j_{\nu}^0(t, \theta(t), u_{\nu}(t); v_{\nu} - u_{\nu}) d\Gamma \quad \text{a.e. on } \Gamma_3.$$

We now combine the equality (6.10) with the latter inequality to get

$$\begin{aligned} \int_{\Omega} \mathbf{u}' \cdot \mathbf{v} d\mathbf{x} + \int_{\Omega} \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) d\mathbf{x} + \int_{\Gamma_3} j_{\nu}^0(t, \theta(t), u_{\nu}(t); v_{\nu} - u_{\nu}) d\Gamma \\ \geq \int_{\Omega} \mathbf{f}_0(t, \theta(t)) \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot (\mathbf{v} - \mathbf{u}) d\Gamma. \end{aligned} \quad (6.11)$$

Finally, we substitute the constitutive law (6.1) in (6.11) to obtain the following variational formulation of the system (6.1)–(6.5).

Find a displacement field $\mathbf{u} \in U$ such that

$$\begin{aligned} \int_{\Omega} \mathbf{u}' \cdot \mathbf{v} d\mathbf{x} + \int_{\Omega} \mathcal{F}(t, \boldsymbol{\varepsilon}(\mathbf{u})) : (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u})) d\mathbf{x} + \int_{\Gamma_3} j_{\nu}^0(t, \theta(t), u_{\nu}(t); v_{\nu} - u_{\nu}) d\Gamma \\ \geq \int_{\Omega} \mathbf{f}_0(t, \theta(t)) \cdot (\mathbf{v} - \mathbf{u}) d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot (\mathbf{v} - \mathbf{u}) d\Gamma, \end{aligned} \quad (6.12)$$

for all $\mathbf{v} \in U$.

We define that

$$\begin{aligned}
 B : \mathbb{R} \times U &\rightarrow U^*; \\
 \langle B(t, \mathbf{u}), \mathbf{v} \rangle &= \int_{\Omega} \mathcal{F}(t, \varepsilon(\mathbf{u})) : \varepsilon(\mathbf{v}) d\mathbf{x}. \\
 J : I \times X \times E &\rightarrow \mathbb{R}, \\
 J(t, y, \mathbf{u}) &= \int_{\Gamma_3} j_{\nu}(\mathbf{x}, y(\mathbf{x}), u_{\nu}(\mathbf{x})) d\mathbf{x}; \\
 f &\in U^*, \quad \langle f, \mathbf{v} \rangle = \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} d\Gamma; \\
 F : \mathbb{R} \times X \times U &\rightarrow X; \\
 F(t, y, \mathbf{u})(\mathbf{x}) &= f(\mathbf{x}, t, y(\mathbf{x}), \mathbf{u}(\mathbf{x})), \\
 R : L^2(0, b; L^2(\Omega)) &\rightarrow L^2(0, b; L^2(\Omega)); \\
 (R\theta)(t) &= \chi \int_0^t \theta(s) ds, \theta \in L^2(0, b; L^2(\Omega)). \\
 M : H &\rightarrow E \quad \text{is the trace operator with the norm } \|M\|, \\
 g : \mathbb{R} \times X \times H &\rightarrow H, \\
 \langle g(t, y, \mathbf{u}), \mathbf{v} \rangle &= \int_{\Omega} \mathbf{f}_0(t, y(t)) \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} d\Gamma, t \in \mathbb{R}, y \in X, \mathbf{u}, \mathbf{v} \in H.
 \end{aligned}$$

Then, we can verify that the abstract hypotheses $H(B)$, $H(J)$, $H(M)$, $H(g)$, (H_b^*) , $H(A^*)$ and $H(R)$ hold with the following constants:

$$\begin{aligned}
 m_F^* &= \sqrt{2}\nu_1, \quad n_F^* = \sqrt{2}\nu_2, \\
 \beta_B &= m_{\mathcal{F}}, \alpha_B = L_{\mathcal{F}}, \\
 \alpha_J &= \kappa_2, \beta_J = \beta_1, \\
 \alpha_g &= \alpha_1 + \alpha_2, \\
 c_R &= \chi.
 \end{aligned}$$

It is trivial to check the conditions $H(M)$, H_b^* and $H(R)$. The hypotheses $H(J)$, $H(M)$, $H(g)$ could be testified by some standard computation similar to [17,20]. For $H(F)$, we have the following estimates:

$$\begin{aligned}
 \|F(t, y, \mathbf{u}) - F(t, z, \mathbf{v})\|_X &= \left(\int_{\Omega} |F(t, y, \mathbf{u})(\mathbf{x}) - F(t, z, \mathbf{v})(\mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\
 &= \left(\int_{\Omega} |f(\mathbf{x}, t, y(\mathbf{x}), \mathbf{u}(\mathbf{x})) - f(\mathbf{x}, t, z(\mathbf{x}), \mathbf{v}(\mathbf{x}))|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\
 &\leq \left(\int_{\Omega} \nu_1 |y(\mathbf{x}) - z(\mathbf{x})| + \nu_2 \|\mathbf{u}(\mathbf{x}) - \mathbf{v}(\mathbf{x})\|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\
 &\leq \sqrt{2}\nu_1 \|y - z\|_X + \sqrt{2}\nu_2 \|\mathbf{u} - \mathbf{v}\|_H.
 \end{aligned}$$

This fact together with the continuity of f lead that all conditions in $H(F)$ have been verified.

By using Theorem 5.1, we have the existence result of periodic solution to (6.1)–(6.9) given by the following theorem.

Theorem 6.1. Suppose that $(\mathcal{H}_{\mathcal{F}})$, $(\mathcal{H}_f)$, $(\mathcal{H}_{j_\nu})$, $(\mathcal{H}_f^*)$ hold. Then the problem (6.1)–(6.9) has a unique mild solution provided that:

$$m_{\mathcal{F}} > \max \{ (\beta_1 \|M\|^2 + \alpha_1 + \alpha_2) \ell^2; (\kappa_2 \|M\|^2 + \alpha_1 + \alpha_2) \ell^2 \},$$

$$e^{-\vartheta b} + \frac{\kappa(\vartheta + \kappa e^{\kappa b} - \vartheta e^{-\vartheta b} - \kappa)}{\vartheta + \kappa} < 1,$$

where

$$\kappa := \max \left\{ \sqrt{2}\nu_1 + b\chi + \|M\|\beta_1 + \alpha_1 + \alpha_2; \sqrt{2}\nu_2 \right\},$$

$$\vartheta := \max \left\{ \frac{m_{\mathcal{F}}}{\ell^2} - (\beta_1 \|M\|^2 + \alpha_1 + \alpha_2); \gamma_1 \right\}.$$

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