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R-C Ladder Design: A Tutorial

Abstract

Passive ladder networks are used in many applications for signal processing such as anti-aliasing filtering, ripple reduction in power networks, and as loop filters in micro-power delta-sigma analog-to-digital converters. In this tutorial, we provide an update on the theory and design of some commonly used passive R-C ladders. The discussion starts with the description of the transfer functions of two-ports, followed by the analysis of a cascade of two-ports, including the special case of identical two-ports. We discuss the estimation of thermal noise in passive ladders. Finally, a simple example is used to illustrate the application of the theory to the design of an R-C ladder with a prescribed transfer function and signal-to-noise ratio.

Index Terms—R-C ladder filters, delta-sigma A/D converters, Cayley-Hamilton theorem, Bashkow's method, noise PSD.

I. Introduction

The design of passive two-ports is extensively discussed in textbooks devoted to circuit theory and filter synthesis [1]. However, the practical aspects of the design process are seldom explored. Since a special class of such circuits, passive R-C ladder networks, is used in many applications, including the implementation of analog integrated circuits [2], [3], [4], it seems that a systematic discussion of their theory and design may be of use. This paper describes the fundamentals of the analysis and design of cascaded two-ports, and then provides a focused discussion of R-C ladder networks for a specified gain response and signal-to-noise ratio. The process is illustrated by a simple design example.

Section II of the paper discusses the chain matrix, which is the convenient tool for the analysis of cascaded two-ports. The special case of the cascade of identical two-ports, which is useful, for example, for the modeling of on-chip interconnects, is separately analyzed. Section III discusses the analysis of signals and noise in R-C ladder filters, and Section IV describes the design of a simple filter for specified bandwidth and signal-to-noise ratio.

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II. The Analysis of Cascaded Two-Ports

Most systems used in communications and electronics can be modeled as a cascade of two-port networks (Figure 1). The relations between the port voltages and currents of such signal chains may be represented by the chain parameters A , B , C , and D [1]. They are defined by the equations:

$$\begin{aligned} V_1 &= AV_2 + BI_2 \\ I_1 &= CV_2 + DI_2 \end{aligned} \quad (1)$$

Here, V_i and I_j are the port voltages and currents. The 2×2 matrix containing the chain parameters is the chain matrix T :

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (2)$$

It follows from the definition of the chain matrix that for the cascaded networks shown in Figure 1, the chain matrix of the complete network is given by the product of the matrices of all two-ports in the chain:

$$T = T_1 T_2 T_3 \dots \quad (3)$$

In an important special case, all two-ports are identical. This situation occurs, for example, in delay lines realized with cascaded lumped all-pass stages, or in the modeling of on-chip interconnects in a CMOS device [2]. Then eq. (3) gives:

$$T = T^N \quad (4)$$

Here, N is the number of cascaded stages. There are computer tools for the calculation of the power of a matrix [5]. However, for lower values of N , it is also possible to use analytic techniques, as described next.

The analytic calculation is based on the characteristic polynomial $P(x)$ of the matrix T [5], which is given

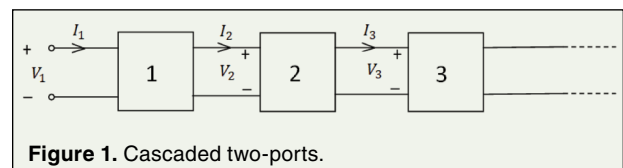


Figure 1. Cascaded two-ports.

by the determinant of the matrix $\{xI_2 - T\}$. Here, I_2 is the 2×2 unit matrix. This results in the relationship:

$$P(x) = \det\{xI_2 - T\} = x^2 + c_1x + c_0 \quad (5)$$

The Cayley-Hamilton Theorem [6] states that if the matrix T replaces x in its own characteristic polynomial, the result is the 2×2 zero matrix 0_2 . This gives the relation:

$$T^2 + c_1T + c_0I_2 = 0_2 \quad (6)$$

From eq. (6),

$$T^2 = (A + D)T - (AD - BC)I_2 \quad (7)$$

Here, I_2 is the 2×2 unit matrix. For reciprocal two-ports $AD - BC = 1$, which allows simplifying the formula (7) into:

$$T^2 = (A + D)T - I_2 \quad (8)$$

This makes finding T^2 easy. To find T^4 , both sides of eq. (8) may be squared, and then (8) applied again to replace T^2 . Continuing this way, all even powers of T may be found. Odd powers will require a single multiplication by T .

As an example, consider the RC stage shown in Figure 2. Here,

$$A = sR_1C_1 + 1, \quad B = R_1, \quad C = sC_1, \quad \text{and} \quad D = 1 \quad (9)$$

Hence, (8) gives the matrix equation:

$$\begin{aligned} T^2 &= (sR_1C_1 + 2)T - I_2 \\ &= \begin{bmatrix} (sR_1C_1)^2 + 3sR_1C_1 + 1 & R_1(sR_1C_1 + 2) \\ sC_1(sR_1C_1 + 2) & sR_1C_1 + 1 \end{bmatrix} \end{aligned} \quad (10)$$

Note that T^2 could have been obtained more simply as $T \cdot T$, but that result would not have been useful for

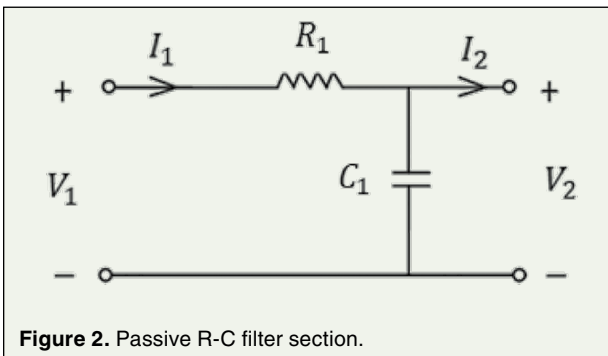


Figure 2. Passive R-C filter section.

finding T^6 or T^8 , whereas eq. (10) is helpful for those tasks also. Note also that T^N may also be found using the eigenvalues and eigenvectors of T .

III. Passive R-C Ladder Filters

An important application of the theory discussed above is for the analysis and design of R-C ladder filters. These circuits can provide a simple way for suppressing out-of-band noise. They are highly linear, and don't require bias power. They are therefore useful as anti-aliasing filters, as smoothing filters at the input and feedback branches of $\Delta\Sigma$ A/D converters, as loop filters of micro-power $\Delta\Sigma$ ADCs [3], and ripple reduction filters in power networks. Figure 3 shows a recent version used in a continuous time $\Delta\Sigma$ A/D converter [4].

To illustrate the design process, we shall next describe the detailed design of a second-order R-C low-pass filter. The technique can be easily generalized and applied to higher-order filters.

An important aspect of the design concerns the thermal noise generated in the circuit. In a passive R-L-C filter the noise is generated only by the resistors. The noise introduced by a resistor R may be represented by a series noise voltage source with a power spectral density (PSD) $V_n^2(f) = 4kTR$, where k is the Boltzmann constant and T is the absolute temperature in Kelvin degrees. In a circuit with several resistors, the noise sources are uncorrelated, and hence the overall output noise PSD can be found by superposition of powers, by adding the PSDs coming from the individual sources. However, in many cases there are more efficient ways of finding the output noise of a passive circuit. If only the variance of the noise voltage V_n^2 (the integral of the noise PSD over all frequencies) is of interest, it can be estimated simply by using Bode's brilliant Noise Theorem [7]. It states that

$$V_n^2 = kT \left(\frac{1}{C_\infty} - \frac{1}{C_0} \right) \quad (11)$$

Here, C_∞ is the capacitance seen at the output port with all resistors open-circuited, and C_0 is the output capacitance when all inductors and resistors are short-circuited.

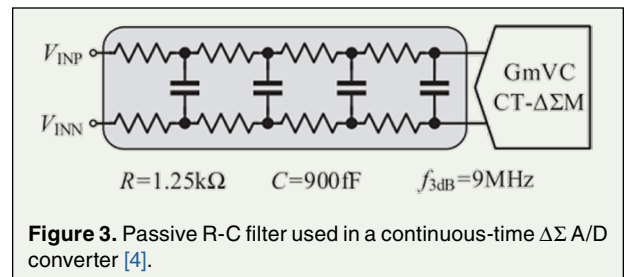


Figure 3. Passive R-C filter used in a continuous-time $\Delta\Sigma$ A/D converter [4].

Eq. (11) doesn't give any information on the PSD frequency response $V_n^2(f)$ of the noise. However, this can also be obtained efficiently using the inter-reciprocity concept [8]. This allows replacing all noise sources associated with the resistors in the actual circuit with a single source connected to the output port of the inter-reciprocal circuit. For a passive network this circuit is the same as the physical circuit. This approach leads to the simple formula for the power spectral density of the output noise voltage of an RLC one-port

$$V_n^2 = 4kT\text{Re}\{Z(j\omega)\} \quad (12)$$

Here, $Z(j\omega)$ is the output impedance of the one-port (Figure 4a). The formula assumes a single-sided spectral distribution. For the calculation of the output noise current, using the Helmholtz/Thévenin theorem (Figure 4b), the relation

$$I_n^2 = 4kT\text{Re}\{Y(j\omega)\} \quad (13)$$

can be obtained.

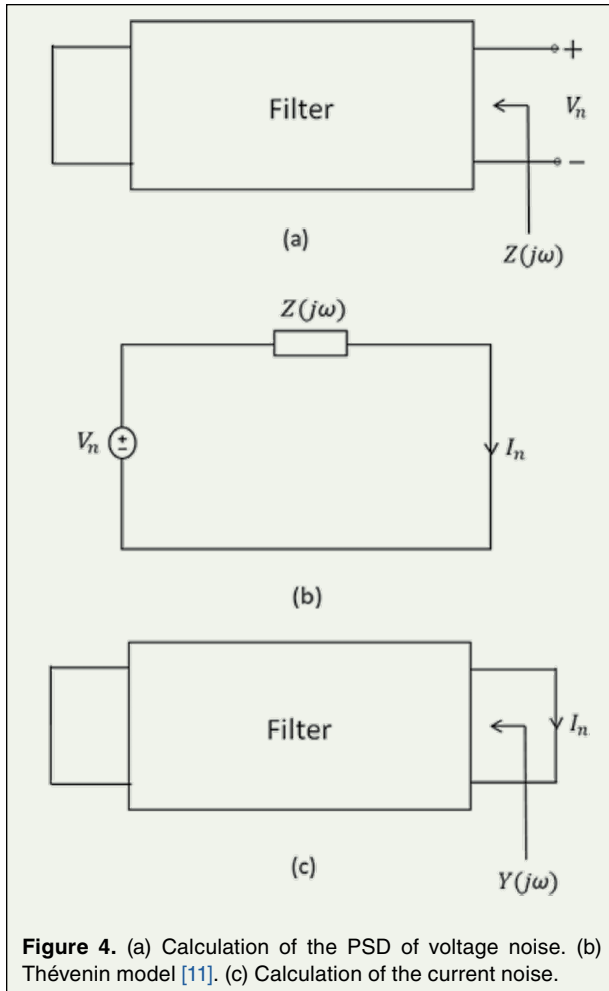


Figure 4. (a) Calculation of the PSD of voltage noise. (b) Thévenin model [11]. (c) Calculation of the current noise.

IV. R-C Filter Example

As an example of the design procedure, the design of the circuit shown in Figure 5 is discussed next. It is assumed that it is driven by a voltage source V_{in} and its output is a current I_o into the virtual ground of a following active stage. This will be the case for example if the filter is connected to the input of an amplifier or of an active R-C integrator.

Single-ended circuit will be discussed here, although a differential implementation (such as shown in Figure 3) may be preferred in many applications. The hand analysis of the simple circuit shown is easy. For higher-order ladders, the efficient analysis technique proposed by Bashkow [9] should be used. In Bashkow's method the output voltage (current) is assumed to be 1 V (1 A), and the corresponding input source voltage (current) is found. Since the transfer function is the same in the two situations, the actual output is easily found. In our circuit, we can assume that $I_o = 1$ A, and find V'_{in} . Then the real value of the output current is $I_o = V_{in}/V'_{in}$.

Note that if the ladder has a repetitive structure where all resistors and capacitors have the same value then efficient analysis may use the derivations given above for the power of the chain matrix.

The structure of the circuit of Figure 5 reveals that its transfer function has two zeros at infinite frequency and two negative real poles. Hence it is of the form

$$Y(s) = \frac{I_o(s)}{V_{in}(s)} = \frac{K}{(s - p_1)(s - p_2)} \quad (14)$$

Here, p_1 and p_2 are negative real numbers. Since the dc gain by inspection is $Y(0) = 1/(R_1 + R_2 + R_3)$, the constant factor is $K = p_1 p_2 / (R_1 + R_2 + R_3)$. The poles may be found by analyzing the circuit. We assume that all resistors have the same value R . The circuit can easily be analyzed using Bashkow's ladder analysis method [9]. Alternatively, the cascaded two-port analysis discussed in Section II may be used. Eq. (10) gave the chain matrix of two cascaded R-C two-ports. The circuit of Figure 5 can be regarded as the cascade of two such stages, terminated by an additional resistor R . Hence, denoting the

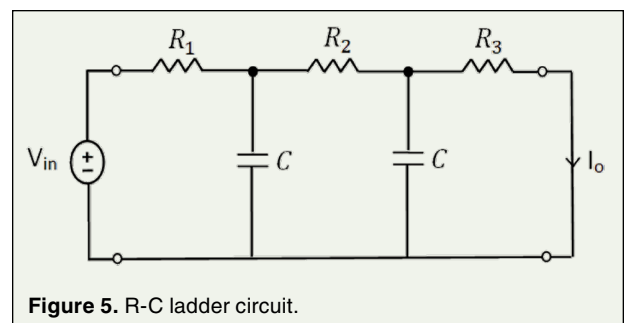


Figure 5. R-C ladder circuit.

chain parameters of the cascade by A' , B' , C' and D' , the equation $V_1 = V_{in} = A'V_2 - B'I_2$ holds. Since $I_2 = -I_{out}$ and $V_2 = -RI_2$, the transfer admittance becomes

$$Y(s) = \frac{I_o(s)}{V_{in}(s)} = \frac{1}{R(s'^2 + 4s' + 3)} \quad (15)$$

where $s' = sRC$. The poles are the zeros of the denominator: $p_1 = -1/RC$ and $p_2 = -3/RC$.

At a frequency ω , the variance of $Y(j\omega)$

$$|Y(j\omega)|^2 = \frac{K^2}{(p_1^2 + \omega^2)(p_2^2 + \omega^2)} \quad (16)$$

The -3 dB signal gain occurs at the frequency where the value of the denominator of $|Y(j\omega)|^2$ become twice its dc value, or $2p_1^2p_2^2$. This gives

$$\omega_3 \approx \frac{0.912}{RC} \quad (17)$$

The -6 dB radian frequency is where the denominator equals $4p_1^2p_2^2$. This leads to

$$\omega_6 \approx \frac{1.487}{RC} \quad (18)$$

In the stopband, the -40 dB frequency occurs where the denominator equals $10^4 p_1^2p_2^2$. This leads to

$$\omega_{40} \approx \frac{17.2}{RC} \quad (19)$$

Near DC, where $\omega \ll |p_1|$, the transfer admittance can be approximated by

$$Y(j\omega) \approx \frac{K}{R(p_1p_2 - j\omega(p_1 + p_2))} \quad (20)$$

Hence, the low-frequency phase shift P is

$$P = \angle Y(j\omega) \approx -\frac{4\omega}{3} \quad (21)$$

and the group delay near $\omega = 0$ is

$$T = \frac{4}{3}RC \approx 1.33RC \quad (22)$$

These values are important parameters if the filter is in a feedback loop.

The thermal noise in the output current can be estimated using eq. (13). Surprisingly, while the power of the signal current is a monotone decreasing function of ω , the noise power rises with ω . This can be predicted from eq. (13). For low frequencies $\text{Re}\{Y(j\omega)\} \approx 1/R$, while for $\omega \gg \omega_6$ $\text{Re}\{Y(j\omega)\} \approx 3/R$, representing a noise increase of about 4.8 dB.

At dc, for equal resistors the noise PSD has the value

$$I_n^2 = \frac{4kT}{R_1 + R_2 + R_3} = \frac{4kT}{3R} \quad (23)$$

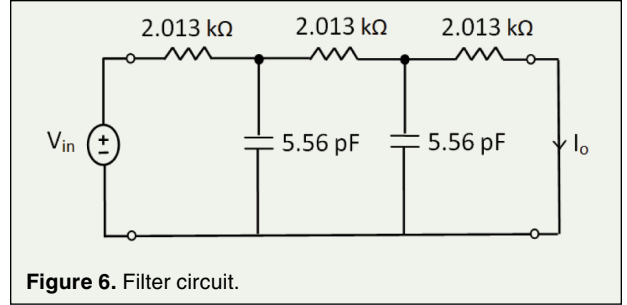


Figure 6. Filter circuit.

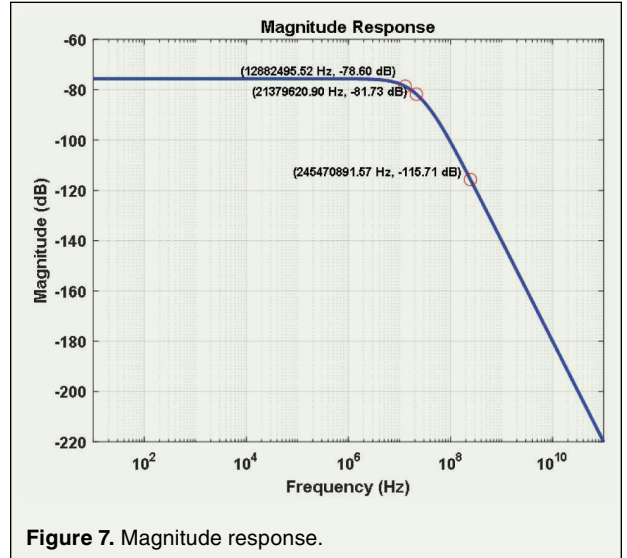


Figure 7. Magnitude response.

where $R_1 = R_2 = R_3 = R$. The in-band noise power is approximately $I_n^2(f)f_B$, where f_B is the signal bandwidth.

The low-frequency signal-to-thermal noise power ratio is therefore

$$\text{STNR} = 10\log_{10}\left(\frac{V_{in}^2}{12kTf_B R}\right) \quad (24)$$

The design of the example filter for a specified input voltage V_{in} , for a specified signal bandwidth $\omega_6 \approx 2.11/RC$ and a specified STNR may be performed in the following steps:

A. We find R from the STNR relation eq. (24);

B. Find C from eq. (18): $C \approx 2.11/3R\omega_6$.

Step A: Assume the specifications $V_{in} = 1$ V, $f_6 = 10$ MHz, $\text{STNR} > 90$ dB. At room temperature, $4kT = 16.56 \times 10^{-21}$ J.

Substituting the specified values into eq. (24) gives $3R \approx 6.04$ kΩ.

Step B: From eq. (18), $C \approx 2.11/6\pi f_6 R \approx 5.56$ pF.

Figure 6 shows the circuit diagram of the filter. Figure 7 illustrates the computed magnitude response of the circuit. Figure 8 illustrates the computed power spectra of the signal and noise of the circuit, verifying the theoretical results.

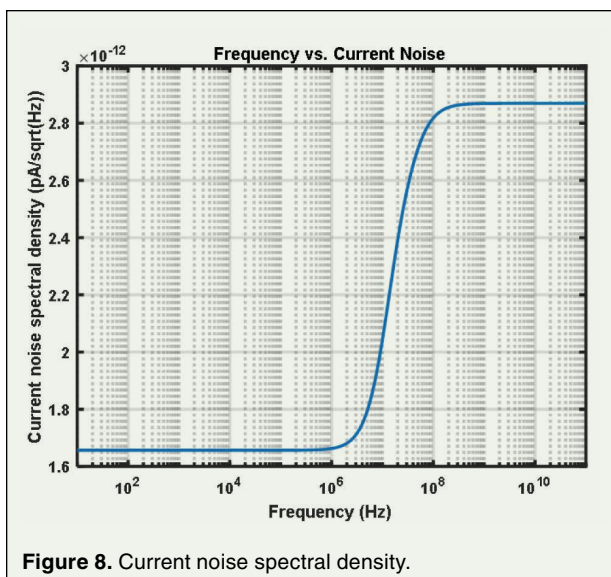


Figure 8. Current noise spectral density.

V. Conclusion

This paper discussed the basic theory of ladder network analysis and design, with special emphasis on the design of R-C ladders. Much of the processes covered can be replaced by using computer-aided design, by MATLAB or Cadence software. Doing so, however, would not provide insight into the physical processes taking place in the circuit.

Future research into this topic may explore the analysis and design of ladders containing lumped as well as distributed components, and/or continuous-time as well as sampled-data stages. The latter topic would be of interest in hybrid noise-shaping analog-to-digital converters [10].



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