

The College Mathematics Journal

ISSN: 0746-8342 (Print) 1931-1346 (Online) Journal homepage: www.tandfonline.com/journals/ucmj20

A Novel Derivation of the Gregory-Leibniz Series for Pi

J. A. Grzesik

To cite this article: J. A. Grzesik (2024) A Novel Derivation of the Gregory-Leibniz Series for Pi, The College Mathematics Journal, 55:4, 346-348, DOI: 10.1080/07468342.2023.2286167

To link to this article: https://doi.org/10.1080/07468342.2023.2286167

© 2023 The Author(s). Published with license by Taylor & Francis Group, LLC.

Published online: 13 Dec 2023.

Submit your article to this journal

Article views: 1643

View related articles

View Crossmark data

Citing articles: 1 View citing articles

Full Terms & Conditions of access and use can be found at
https://www.tandfonline.com/action/journalInformation?journalCode=ucmj20

CLASSROOM CAPSULES

EDITORS
Ricardo Alfaro
University of Michigan-Flint
ralfaro@umflint.edu
Lixing Han
University of Michigan-Flint
lxhan@umflint.edu
Kenneth Schilling
University of Michigan-Flint
ksch@umflint.edu
Livia Hummel
University of Indianapolis
hummell@uindy.edu

Classroom Capsules are short (1–3 pages) notes that contain new mathematical insights on a topic from undergraduate mathematics, preferably something that can be directly introduced into a college classroom as an effective teaching strategy or tool. Classroom Capsules should be prepared according to the guidelines on the inside front cover and submitted through Editorial Manager.

A Novel Derivation of the Gregory-Leibniz Series for Pi

J. A. Grzesik (jan.grzesik@hotmail.com) Allwave Corp., 3860 Del Amo Boulevard, Suite #404, Torrance, CA 90503

Most of us have encountered the celebrated series, variously ascribed to Gregory and Leibniz,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots, \tag{1}$$

as being implied by an arctangent evaluation in accordance with

$$\begin{aligned} \arctan(x) &= \int_0^x \frac{d\zeta}{1+\zeta^2} \\ &= \int_0^x \left\{1 - \zeta^2 + \zeta^4 - \zeta^6 + \zeta^8 - \zeta^{10} + \cdots\right\} d\zeta \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots, \end{aligned} \tag{2}$$

whereupon setting $x = 1$ yields (1).

Discovered in the late 17th century by James Gregory, series (1) was contemporaneously rediscovered by Gottfried Wilhelm von Leibniz. Leibniz arrived at (1) by ingenious geometric reasoning aimed at recovering the $\pi/4$ area of a unit radius circular quadrant. Albeit longer than the essentially zinger-like simplicity of (2), Leibniz's proof is richly deserving of attention by virtue of its brilliance. Accounts can be found in [**1**]–[**3**], and doubtless elsewhere. Leibniz himself held his proof in high esteem, and one hears the same enthusiasm in the concluding exclamation "WOW!" whereby Frank Burk, the author of [**1**] (see pp. 12–14), proclaims that he, too, admires. Not to be outdone, George Simmons, the author of [**3**] (see pp. 264–266), echoes an identical sentiment, although in more muted terms. Wikipedia entries under the respective headings of *Gregory's Series* and *Madhava of Sangamagrama* further assert that both series (1) and (2) had already been known to an Indian mathematician named Madhava, whose life straddled much of the fourteenth and the early part of the fifteenth

doi.org/10.1080/07468342.2023.2286167

This is an Open Access article distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives License (http://creativecommons.org/licenses/by-nc-nd/4.0/), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the original work is properly cited, and is not altered, transformed, or built upon in any way. The terms on which this article has been published allow the posting of the Accepted Manuscript in a repository by the author(s) or with their consent.

 © 2023 THE AUTHOR(S). PUBLISHED WITH LICENSE BY TAYLOR & FRANCIS

centuries. One is astonished to realize that, in all likelihood, Madhava arrived at such achievements without being able to benefit from the facile methods of the calculus yet to be developed some three centuries hence in far-off Europe.

We ourselves quite unexpectedly discovered a fresh route leading to (1) during one of several attempts to tackle an *American Mathematical Monthly* problem. These efforts had led us to consider, for any positive integer m, the integral

$$\int_0^{\pi/4} \tan^{2m}(\varphi)\, d\varphi$$

which, because $\tan^2(\varphi) = \sec^2(\varphi) - 1$ whereas $d\tan(\varphi)/d\varphi = \sec^2(\varphi)$, submitted at once to a closed-form quadrature in accordance with a downward recurrence diminishing by two at each step the surviving power of the tangent. And so, since the initial power of $2m$ was thus ultimately brought to zero, the final term of the recurrence, apart from sign,

$$\int_0^{\pi/4} d\varphi = \frac{\pi}{4},$$

thrust into the open precisely the $\pi/4$ ratio now sought. Moreover, because $0 \le \tan(\varphi) < 1$ throughout $0 \le \varphi < \pi/4$, one could reasonably expect the sequence of integrals $\int_0^{\pi/4} \tan^{2m}(\varphi) d\varphi$ to dwindle endlessly with unbounded growth in power $2m$. A mechanism was thus revealed for suggesting $\pi/4$ as the limit of an infinite, convergent series of alternating terms.

Convergent quadrature sequence and its downward recurrence. We can follow the progress of the downward recurrence by writing its two early stages as

$$\begin{aligned}
\int_0^{\pi/4} \tan^{2m}(\varphi)\, d\varphi &= \int_0^{\pi/4} \tan^{2(m-1)}(\varphi) \sec^2(\varphi)\, d\varphi - \int_0^{\pi/4} \tan^{2(m-1)}(\varphi)\, d\varphi \\
&= \int_0^{\pi/4} \tan^{2(m-1)}(\varphi) \frac{d\tan(\varphi)}{d\varphi}\, d\varphi - \int_0^{\pi/4} \tan^{2(m-1)}(\varphi)\, d\varphi \\
&= \frac{1}{2m-1} \tan^{2m-1}(\varphi)\Big|_{\varphi=0}^{\varphi=\pi/4} - \int_0^{\pi/4} \tan^{2(m-1)}(\varphi)\, d\varphi \qquad (3) \\
&= \frac{1}{2m-1} - \int_0^{\pi/4} \tan^{2(m-1)}(\varphi)\, d\varphi \\
&= \frac{1}{2m-1} - \frac{1}{2m-3} + \int_0^{\pi/4} \tan^{2(m-2)}(\varphi)\, d\varphi\,.
\end{aligned}$$

Continuing on in this way, we finally arrive after m steps at

$$\int_0^{\pi/4} \tan^{2m}(\varphi)\, d\varphi = \frac{1}{2m-1} - \frac{1}{2m-3} + \cdots + (-1)^{m-1} - (-1)^{m-1}\frac{\pi}{4}, \qquad (4)$$

a result more succinctly stated as

$$\int_0^{\pi/4} \tan^{2m}(\varphi)\, d\varphi = (-1)^{m-1} \left\{ \sum_{n=1}^{m} \frac{(-1)^{n-1}}{2n-1} - \frac{\pi}{4} \right\} \qquad (5)$$

by listing its first m terms in reverse order. At this point, we can undertake to pass to the limit as $m \to \infty$ and, should the anticipation that indeed

$$\int_0^{\pi/4} \tan^{2m}(\varphi)\, d\varphi \underset{m \to \infty}{\longrightarrow} 0 \tag{6}$$

be verified, the Gregory-Leibniz series will have emerged as an automatic consequence.

As it happens, a transparent proof of (6) is already to be found in the fourth line of (3), which permits us to write

$$0 < 2\int_0^{\pi/4} \tan^{2m}(\varphi)\, d\varphi < \int_0^{\pi/4} \tan^{2m}(\varphi)\, d\varphi + \int_0^{\pi/4} \tan^{2(m-1)}(\varphi)\, d\varphi = \frac{1}{2m-1} \tag{7}$$

and thus to encounter a pincer squeeze as $m \to \infty$, validating both existence and value of the limit conjectured in (6), and, of course, series limit (1) as outcome.

Still another elementary, slightly more geometric proof of (6) can be fashioned on the basis of the upward concavity on $0 \le \varphi < \pi/2$ of $\tan(\varphi)$. This alternate proof is longer, and it is rendered more intricate by the need to accommodate the nonuniformity against argument φ of the rate at which $\tan^{2m}(\varphi)$ declines to zero. And, at an even greater remove from elementary reasoning, one which, admittedly, does take nonuniform convergence in easy stride, we can simply invoke the Lebesgue Dominated Convergence Theorem, whose applicability here is easily verified. In the context, however, such level of abstraction lacks the immediacy of conviction which attaches to the explicit value confinement conveyed by (7).

In closing, one cannot help but remark that all proofs of (1) presently mentioned, ours included, do hover, in one way or another, around some attribute of the tangent function. Ours, of course, distinguishes itself by tapping into the dominance by unity of $\tan(\varphi)$ across all of $0 \le \varphi < \pi/4$.

Acknowledgment. The author wishes to thank the editor, Ricardo Alfaro, for encouraging him to provide an instantly apprehended, economical proof that the sequence of tangent integrals on the left in (4) and (5) does indeed converge to zero, and in general in shaping the overall narrative tone.

Summary. We have examined the special sequence of integrals from 0 to $\pi/4$ of even integer powers of the tangent. Each member of this sequence admits an explicit evaluation through a downward recurrence terminating at $\pm\pi/4$. Its very first step embodies an automatic proof of sequence convergence to zero, confirming intuitive anticipation based on the fact that the tangent is strictly less than one when its argument lies interior to $[0, \pi/4)$. As a direct consequence of this convergence, the limiting recurrence provides a fresh access to the historically acclaimed Gregory-Leibniz series for $\pi/4$. It should moreover please the reader to realize that entirely similar reasoning applies to the special sequence having odd tangent powers, in which case one recovers the Newton-Mercator-Hudde alternating harmonic series [**4**] for $\log(2)$. Indeed, in the resulting counterpart to (5), each denominator within the sum advances by one unit, $2n-1 \to 2n$, whereas the final term becomes $\int_0^{\pi/4} \tan(\varphi)\, d\varphi = \log(2)/2$, with an obligatory minus sign still in abeyance.

References

[1] Burk, F. (1998). *Lebesgue Measure and Integration, An Introduction*. New York: Wiley.
[2] Leibniz's Formula for Pi, *Proof Wiki*.
[3] Simmons, G. (1992). *Calculus Gems, Brief Lives of Memorable Mathematicians*. New York: McGraw-Hill.
[4] Mercator series, *Wikipedia*.

 © 2023 THE AUTHOR(S). PUBLISHED WITH LICENSE BY TAYLOR & FRANCIS