



# On bounded variation solutions of generalized fuzzy differential equations

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## Abstract

The fuzzy Henstock integral provides a flexible and powerful tool for handling high oscillation functions, its primitive is a fuzzy bounded variation function rather than a fuzzy absolutely continuous function. In this paper, we first provide some basic properties of fuzzy bounded variation function spaces. Then it was proved that the solution of the generalized fuzzy differential equations (GFDEs) based on fuzzy Henstock integral must be a bounded variation function, which is to give the structure for the bounded variation solutions (BV-solutions) of the GFDEs. Finally, by using the embedding theorem for fuzzy number space and Schauder Tichonov's fixed point theorem, we obtain the existence theorem of BV-solutions for GFDEs. In addition, we provide the local uniqueness theorems for two types of fuzzy solutions of GFDEs.

**Keywords** Generalized fuzzy differential equations · Bounded variation solutions · Existence and uniqueness theorems · Fuzzy number · Fuzzy analysis · Fuzzy Henstock integral

**Mathematics Subject Classification** 03E72 · 26E50 · 34A07 · 34D20

## 1 Introduction

As we all know, there are critical applications in many scientific fields for the differential equations, such as automatic control, design of various electronic devices, trajectory calculation, flight stability of aircraft and missiles, stability of chemical reaction process, etc. However, the solutions of ordinary differential equations must be absolutely continuous functions. To expand the solutions of differential equations, Henstock integral was proposed by

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Henstock (1963) and also independently by Kurzweil (1957), which extended the Riemann integral, improper Riemann integral, Lebesgue integral and Newton integral (1989).

Since Chang and Zadeh (1972) put forward fuzzy numbers, the study of properties for fuzzy numbers (in particular algebraic properties) has attracted many researcher's attention. However, for each fuzzy number pair, the difference is not well defined until g-difference is proposed (see Bede and Stefanini 2013). In fact, the fuzzy derivative and the subtraction of fuzzy numbers are developed simultaneously. Puri and Ralescu (1983) proposed the H-derivative for fuzzy-valued functions using H-difference. This definition is similar to the definition of derivative in real analysis. Due to the limitation of H-derivative, in 2005, Bede and Gal (2005) introduced the strongly generalized derivative. Since then, Bede et al (2007), Stefanini and Bede (2014) have used the strong generalized derivative for further research.

As the applications of the fuzzy derivatives, Kaleva (1987) first researched the following

$$\tilde{x}'(t) = \tilde{f}(t, \tilde{x}(t)), \quad \tilde{x}(t_0) = \tilde{x}_0 \quad (1)$$

where  $I = [a, b]$ ,  $\mathbb{R}_{\mathcal{F}}$  is the space of fuzzy numbers and  $\tilde{f} : I \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$  is fuzzy continuous with  $\tilde{x}_0 \in \mathbb{R}_{\mathcal{F}}$ .

## 1.1 Related work

After then, the existence theorems and uniqueness theorems of the solutions for (1) have been studied by many scholars (Allahviranloo et al 2009; Kaleva 2006; Lakshmikantham and Vatsala 2000) under various conditions, but they have not achieved satisfactory results. We know the solution does not necessarily exist if only based on the continuity of  $\tilde{f}$  for (1). The fuzzy number space  $\mathbb{R}_{\mathcal{F}}$  is not a vector space, but it is isomorphic to a cone in a Banach space (see, Kaleva 1998). Until 1999, Nieto (1999) obtained that problem (1) have a unique solution if  $\tilde{f}$  is bounded and continuous. Using the embedding theorem (Theorem 1), C. Wu and S. Song Wu and Song (1998); Wu et al (1996) translated the certain problem (1) to a ordinary differential equation in a closed convex cone of  $\overline{C}[0, 1] \times \overline{C}[0, 1]$ . Furthermore, they proved that if the generalized Lipschitz conditions is satisfied by  $\tilde{f}$ , the solution for (1) is unique. In 2008, combined with the properties of the Hasegawa function, under dissipative-type conditions, we Gong and Shao (2008) obtained the global existence theorems of the solutions for (1). In 2022, Ghaffari et al (2022) obtained the traveling wave fuzzy solutions for the fuzzy fractional wave equation and the fuzzy fractional diffusion equation by using fuzzy traveling wave method. Bahrami et al (2023) based on the direct and inverse fuzzy transforms derived analytical approximate solution of flexible basic functions for the fractional-order linear semi-explicit differential-algebraic equations with variable coefficients. By using the differential inclusion theory, Dai and Chen (2020) studied the structural stability for periodic solutions of problem (1). More research by Chen et al. for FDEs, see Dai and Chen 2017; Dai et al. 2018; Sin et al. 2017.

In addition, the research on the integro-differential equations and boundary value problem has also achieved good results. By introducing the mixed solutions of the FDEs, Alikhani et al (2012) discussed the existence and uniqueness of global solutions for the FDEs of Volterra type. Using the fixed point theorem, Villamizar-Roa et al (2015) proved the existence and uniqueness theorem of the FDEs. For more in-depth research for fuzzy integro-differential equations (see Liu et al 2020; Qahremani et al 2021). Using mean values and left-right spread systems, Akram et al (2021) obtained the solutions for L R-bipolar fuzzy linear system, L R-complex bipolar fuzzy linear system with real coefficients and L R-complex bipolar fuzzy linear system with complex coefficients of equations in 2021. Using Laplace trans-

form technique, Akram et al (2022) determined a potential solution for linear third-order fuzzy initial-value problem, the solution is represented in terms of the MittagLeffler function involving a single series. By using the fuzzy Laplace transform, Akram et al (2023) obtained the analytical solution of a two-dimensional homogeneous system of fuzzy linear fractional differential equations with the Caputo derivative of two independent fractional orders.

In 2023, Laiate (2023) provides algebraic properties of the  $\psi$ -cross operations on S-linearly correlated fuzzy numbers and study the fuzzy polynomial differential equations under the  $\psi$ -cross product. And apply the results to solve the Abel equation. In 2024, Sarvestani and Chehlabi (2024) obtain some properties of continuity of the derivative function and the existence of switching points for a fuzzy-valued function based on the generalized differentiability. And they prove the new sufficient conditions for existence of solutions for periodic boundary problems of first-order linear fuzzy differential equations. By the finite difference method, Wasques (2024) presents a numerical approach to solve fuzzy partial differential equation restrict to fuzzy boundary and initial conditions. They apply this method to solve the heat equation. Yang and Wu (2024) focus on solve a class of second-order linear fuzzy differential equation semi-homogeneous boundary value problem. They prove the existence and uniqueness of Green function for semi-homogeneous BVP of self-conjugate equations and present several examples to illustrate the solution of boundary value problem for fuzzy differential equations using Green function method.

However, in all the papers mentioned above, the right-hand side function of the FDEs is Kaleva integrable. In 2002, Xue and Fu (2002) obtained the Carathéodory solutions for the FDEs. If  $\hat{f}$  is a highly oscillatory function, then the solution of the FDEs is a bounded variation function. Therefore, it is not enough to consider only  $\hat{f}$  satisfies Carathéodory condition and  $\hat{f}$  is a Kaleva integrable function. In 2000, Wu and Gong (2000, 2001) gave the concept of fuzzy Henstock integral, which is a nonabsolute fuzzy integral. To complement fuzzy Henstock integral theory, for the fuzzy Henstock integrals, Shao et al (2017, 2019) studied some controlled convergence theorems. As its applications, they considered the retarded fuzzy functional differential equations, and obtained the existence and uniqueness theorem of the solutions.

## 1.2 Motivation

To sum up, it is not enough to consider only Carathéodory solutions in Xue and Fu (2002), and it is also not enough to consider only the Kaleva integral. Our motivation and innovations can be summarized as follows:

1. Although a large number of studies have been committed to expanding the solutions of fuzzy differential equations, most of the existing studies of fuzzy differential equations are fuzzy absolute continuous function. In the case of fuzzy Henstock integrals, the solution of the GFDEs is of fuzzy bounded variation. Therefore, to enlarge the class of solutions of fuzzy differential equations, this paper considers the bounded variation solutions of GFDEs based on the fuzzy Henstock integral, and extends the solutions of GFDEs from the fuzzy absolutely continuous function space to the fuzzy bounded variation function space.
2. Before g-difference was proposed, the subtraction of a pair of fuzzy numbers did not always exist, and the main drawback of solving fuzzy differential equations was that the existence theorem always assumed that the right-end function was subtractable. Therefore, this paper uses g-difference to handle the right-end function of GFDEs, making the

conclusions obtained in this paper more extensive than those based on H-difference and gH-difference.

Expanding the space of solutions for fuzzy differential equations is meaningful, as it can lead to some fuzzy differential equations having solutions in a broader sense, even though there were no solutions in previous sense, allowing for further research. The original fuzzy differential equations studied the properties of solutions in the fuzzy absolutely continuous function space. However, the primitive function of the fuzzy Henstock integral is in the fuzzy bounded variation function space, which includes the fuzzy absolutely continuous function space. In theory, we have the foundation to study the solutions of fuzzy differential equations based on fuzzy Henstock integrals in fuzzy bounded variation function spaces. To ensure that 1. The solution of a fuzzy differential equation must be a fuzzy bounded variation function and 2. any bounded variation function must be fuzzy Henstock integrable, we need to impose certain restrictions on the integrand (Definition 10). These limitations ensure that the integrand function can be controlled, but they also require that any two fuzzy numbers can be subtracted, which is the property that only possessed by g-difference. It is easy to think of extending the derivative of fuzzy differential equations from the gH-derivative to the g-derivative to expand the solution space of fuzzy differential equations. Therefore, this paper extends the solution space of fuzzy differential equations from the fuzzy absolute continuous function space to the fuzzy bounded variation function space based on the fuzzy Henstock integral and the g-difference between fuzzy numbers.

The study of BV-solutions for GFDEs is meaningful. In fact, there exists a generalized fuzzy differential equation where the right-hand function is not integrable in the sense of Kaleva integral due to its discontinuity, but there is a solution. Let's use the following examples to show such a generalized fuzzy differential equation.

**Example 1** Considering the GFDEs as follows,  $I = [a, b]$ ,  $\tilde{x}_0 = \tilde{A} \in \mathbb{R}_{\mathcal{F}}$  with

$$\tilde{A}(s) = \begin{cases} s, & 0 \leq s \leq 1 \\ 2-s, & 1 < s < 2 \\ 0, & \text{others.} \end{cases}$$

let  $\tilde{f}(t) = \chi_{g(t)} + \tilde{A}$  where  $\chi$  transforms  $g$  into a fuzzy-number-valued function such that  $[\chi_{g(t)}]_{\lambda} = [g(t), g(t)]$  for  $\forall \lambda \in [0, 1]$ , and  $g : [a, b] \rightarrow \mathbb{R}$  is a real-valued function as follows:

$$g(t) = \begin{cases} 2t \sin \frac{1}{t^2} - \frac{2}{t} \cos \frac{1}{t^2}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

The above fuzzy differential equation has the following solution:  $\tilde{x}(t) = \chi_{G(t)} + \tilde{A}t$ , where

$$G(t) = \begin{cases} t^2 \sin \frac{1}{t^2}, & t \neq 0 \\ 0, & t = 0. \end{cases}$$

$\tilde{f}(t)$  is not integrable in the sense of Kaleva integral, the method of using integral equations to solve the above fuzzy differential equations is limited.

We will illustrate in Example 5 that the above differential equations no (i) and (ii) solutions in the sense of gH-differential, but there exists a bounded variation solutions in the sense of g-differential.

### 1.3 Outline of the article

The essence of the first part of this article is to derive some basic properties including the relevant theories of fuzzy bounded variation function spaces, and the connection between solutions of GFDEs and fuzzy bounded variation functions, which provide theoretical support for the study of bounded variation solutions for GFDEs. Further, based on the embedded theory of fuzzy number space and Schader Tichonov's fixed point theorem, we prove the existence and uniqueness theorems of GFDEs. We summarize the main contributions of our research work as follows:

1. Proving the existence and uniqueness of bounded variation solutions for GFDEs based on fuzzy Henstock integrals.
2. Using the g-difference between fuzzy numbers to prove the existence and uniqueness of GFDEs.
3. Extend the solutions of GFDEs from the fuzzy absolutely continuous function space to the fuzzy bounded variation function space.
4. Provide specific examples to demonstrate how to use the main results of this paper to obtain the existence and uniqueness of bounded variation solutions.

### 1.4 Layout of the paper

This paper is rest: In Sect. 2, we provide some basic notion and preliminary results. In Sect. 3, we study the structure of the BV-solutions for the FDEs. In Sect. 4, we prove the existence theorems of the BV-solutions for the GFDEs. In Sect. 5, we prove the uniqueness theorems of the BV-solutions for the GFDEs. At last, we give some conclusions and future research in Sect. 6.

## 2 Preliminaries

In this section, some preliminary notion are introduced as a prework, including operations and metric of fuzzy numbers and properties of the fuzzy Henstock integrals. Especially, some important results about fuzzy Henstock integrals can be found in Gong 2004; Gong and Shao 2009.

### 2.1 The fuzzy number space

Throughout our paper  $\mathcal{F}(\mathbb{R})$  denotes a set of all fuzzy subsets on  $\mathbb{R}$ . For  $0 < \alpha \leq 1$ , denote  $[\tilde{u}]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\} = [u_\lambda^-, u_\lambda^+]$ . Suppose  $\tilde{u} \in \mathcal{F}(\mathbb{R})$ , satisfies  $\tilde{u}$  is normal, convex, upper semi-continuous and  $[\tilde{u}]_0 = \bigcup_{\alpha \in (0,1]} [\tilde{u}]_\alpha$  is a bounded set, then  $\tilde{u}$  is called a fuzzy number. We use  $\mathbb{R}_{\mathcal{F}}$  to denote the fuzzy number space. For addition and scalar multiplication of fuzzy numbers, refer to Diamond and Kloeden (1994); Kaleva (1987). In the paper, the multiplication of a fuzzy number  $\tilde{A}$  and a real number  $\beta$  is unified as  $\beta\tilde{A}$ .

Define the Hausdorff metric  $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_+ \cup \{0\}$  by

$$D(\tilde{u}, \tilde{v}) = \sup_{\lambda \in [0,1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where  $[\tilde{u}]_\lambda = [u_\lambda^-, u_\lambda^+]$ ,  $[\tilde{v}]_\lambda = [v_\lambda^-, v_\lambda^+]$ . Using the results of Bede and Stefanini (2013):

- (i)  $(\mathbb{R}_{\mathcal{F}}, D)$  is complete;
- (ii)  $D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$  for  $\tilde{u}, \tilde{v}, \tilde{w} \in \mathbb{R}_{\mathcal{F}}$ ;
- (iii)  $D(k\tilde{u}, k\tilde{v}) = |k|D(\tilde{u}, \tilde{v})$ ,  $k \in \mathbb{R}$ .

**Definition 1** (Bede and Stefanini (2013)) The g-difference of  $\tilde{u}, \tilde{v} \in \mathbb{R}_{\mathcal{F}}$  is given as follows

$$[\tilde{u} \ominus_g \tilde{v}]_{\lambda} = cl(conv \bigcup_{\beta \geq \lambda} ([\tilde{u}]_{\beta} \ominus_{gH} [\tilde{v}]_{\beta}))$$

for all  $\lambda \in [0, 1]$ , where  $[\tilde{u}]_{\beta} \ominus_{gH} [\tilde{v}]_{\beta}$  represent the gH-difference between  $[\tilde{u}]_{\beta}$  and  $[\tilde{v}]_{\beta}$ .

Next, we will provide an example of obtaining the g-difference between two fuzzy numbers  $\tilde{u}$  and  $\tilde{v}$ . In this example, the gH-difference between  $\tilde{u}$  and  $\tilde{v}$  does not exist, but the g-difference does exist. This fact precisely demonstrates that g-differential promotes gH-differential.

**Example 2** Let  $\tilde{u}$  is a triangular fuzzy number and  $\tilde{v}$  is a trapezoidal fuzzy number such that  $[\tilde{u}]_{\lambda} = [\lambda, 2 - \lambda]$  and  $[\tilde{v}]_{\lambda} = [0, 2]$ . Therefore,

$$[\tilde{u} \ominus_g \tilde{v}]_{\lambda} = cl(conv \bigcup_{\beta \geq \lambda} ([\lambda, 2 - \lambda] \ominus_{gH} [0, 2])) = cl(conv \bigcup_{\beta \geq \lambda} ([-\lambda, \lambda])) = [-1, 1].$$

That means the g-difference between  $\tilde{u}$  and  $\tilde{v}$  is a trapezoidal fuzzy number with  $[\tilde{u} \ominus_g \tilde{v}]_{\lambda} = [-1, 1]$ .

**Theorem 1** (Wu and Ma (1991)) Denote  $j(\tilde{u}) = (u^-, u^+)$  for  $\tilde{u} \in \mathbb{R}_{\mathcal{F}}$ . Then  $j : \mathbb{R}_{\mathcal{F}} \rightarrow \overline{C}[0, 1] \times \overline{C}[0, 1]$  satisfies that  $j(\mathbb{R}_{\mathcal{F}})$  is a closed convex cone with vertex  $\tilde{0}$  with

- (i)  $j(a\tilde{u} + b\tilde{v}) = aj(\tilde{u}) + bj(\tilde{v})$  for  $a \geq 0, b \geq 0$ ;
- (ii)  $D(\tilde{u}, \tilde{v}) = ||j(\tilde{u}) - j(\tilde{v})||$ .

i.e.  $j$  embeds  $\mathbb{R}_{\mathcal{F}}$  into  $\overline{C}[0, 1] \times \overline{C}[0, 1]$  isometrically and isomorphically.

## 2.2 The fuzzy Henstock integrals

Let  $\delta(t) : [a, b] \rightarrow \mathbb{R}^+$ . Given the partition  $\Gamma = \{[u, v], \xi\}$ , if

- (i)  $a = t_0 < t_1 < \dots < t_n = b$ ;
- (ii)  $\xi_i \in [t_{i-1}, t_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ ,  
then  $\Gamma$  is  $\delta$ -fine (see Lee (1989)). We call  $\xi_i$  the associated point of  $[t_{i-1}, t_i]$  and  $t_i$  the division point.

**Definition 2** (Wu and Gong (2000, 2001)) If there is  $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$ , for  $\forall \varepsilon > 0$ ,  $\exists \delta(\xi) > 0$  s.t. for every  $\Gamma = \{[u, v], \xi\}$  which is  $\delta$ -fine,

$$D\left(\vartheta(\tilde{f}, \Gamma), \tilde{A}\right) < \varepsilon$$

where  $\vartheta(\tilde{f}, \Gamma) = \sum \tilde{f}(\xi)(v - u)$ , then  $\tilde{f}$  is fuzzy Henstock integrable and write  $(FH) \int_a^b \tilde{f}(t)dt = \tilde{A}$  with  $\tilde{f} \in FH[a, b]$ .

**Theorem 2** (Wu and Gong (2001))  $\tilde{f} \in FH[a, b]$  iff for  $\forall \varepsilon > 0$ ,  $\exists \delta(\xi) > 0$  s.t.

$$D\left(\vartheta(\tilde{f}, \Gamma_1), \vartheta(\tilde{f}, \Gamma_2)\right) < \varepsilon$$

for every  $\Gamma_1$  and  $\Gamma_2$  which are  $\delta$ -fine.

**Definition 3** (Bede and Stefanini (2013)) Let  $t_0 \in (a, b)$  and  $\tilde{f} : (a, b) \rightarrow \mathbb{R}_{\mathcal{F}}$ . If there exists  $\tilde{f}'_g(t_0) \in \mathbb{R}_{\mathcal{F}}$ , s.t.

$$\lim_{h \rightarrow 0} \frac{\tilde{f}(t_0 + h) \ominus_g \tilde{f}(t_0)}{h} = \lim_{h \rightarrow 0} \frac{\tilde{f}(t_0) \ominus_g \tilde{f}(t_0 - h)}{h} = \tilde{f}'_g(t_0),$$

then  $\tilde{f}$  is g-differentiable at  $t_0$ .

**Remark 1** In this paper, the g-differential is applied to all problems about differentiable fuzzy-valued functions.

Denote  $\mathcal{X}[a, b]$  by the continuous function space  $f : [a, b] \rightarrow \overline{C}[0, 1] \times \overline{C}[0, 1]$ , then  $\mathcal{X}[a, b]$  is a Banach space with

$$\|\cdot\|_{\mathcal{X}[a, b]} = \sup_{t \in [a, b]} \|f(t)\|_{\overline{C}[0, 1] \times \overline{C}[0, 1]}.$$

**Theorem 3** Let  $j$  be the embedding operator from  $(\mathbb{R}_{\mathcal{F}}, D)$  to  $\overline{C}[0, 1] \times \overline{C}[0, 1]$  and denote  $C([a, b], \mathbb{R}_{\mathcal{F}})$  by the continuous function space  $\tilde{f} : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ , then  $C([a, b], \mathbb{R}_{\mathcal{F}})$  can be regarded as a closed convex subset in  $\mathcal{X}[a, b]$ .

**Proof** If  $\tilde{f}, \tilde{g} \in C([a, b], \mathbb{R}_{\mathcal{F}})$ ,  $\alpha \in [0, 1]$  then  $\alpha \tilde{f} + (1 - \alpha) \tilde{g} \in C([a, b], \mathbb{R}_{\mathcal{F}})$ , that is  $C([a, b], \mathbb{R}_{\mathcal{F}})$  is a convex subset in  $\mathcal{X}[a, b]$ .

Next, let us show that  $C([a, b], \mathbb{R}_{\mathcal{F}})$  is a closed subset of  $\mathcal{X}[a, b]$ . Denote by  $f_n \triangleq j \circ \tilde{f}_n$  and  $f_n(t) = (f_n(t)^-, f_n(t)^+)$ . Let  $\tilde{f}_n \in C([a, b], \mathbb{R}_{\mathcal{F}})$  be a sequence which satisfied that  $f_n(t)$  converges to  $f(t) \in \mathcal{X}[a, b]$ . By the properties of fuzzy numbers and the Definition of  $\mathcal{X}[a, b]$ , it enough to show  $f(t)^-(f(t)^+)$  is bounded and nondecreasing (non-increasing) on  $[0, 1]$ .

Due to

$$\|f_n(t) - f(t)\|_{\overline{C}[0, 1] \times \overline{C}[0, 1]} \leq \|f_n - f\|_{\mathcal{X}[a, b]}$$

for every  $t \in [a, b]$ , and  $\tilde{f}_n \in C([a, b], \mathbb{R}_{\mathcal{F}})$ .  $f_n(t)^-$  is bounded, left continuous and nondecreasing on  $[0, 1]$  that is let  $r_1 < r_2 \in [0, 1]$ ,  $t \in [a, b]$ ,  $f_n(t)^-(r_1) - f_n(t)^-(r_2) \leq 0$ . From  $f_n(t)^- \rightarrow f(t)^-$  for every  $t \in [a, b]$ ,  $f(t)^-(r_1) - f(t)^-(r_2) \leq 0$ . Similarly,  $f(t)^+(r_2) - f(t)^+(r_1) \leq 0$  and  $f(t)^-(1) \leq f(t)^+(1)$ .

Therefore there is a unique  $\tilde{f} \in \mathbb{R}_{\mathcal{F}}$  s.t. for each  $\lambda \in [0, 1]$

$$[\tilde{f}]_{\lambda} = [f(t)^-(\lambda), f(t)^+(\lambda)].$$

□

### 3 The Properties of BV-solutions for GFDEs

In this section, we give some properties of the BV-solutions for GFDEs. Some useful theorems are obtained by using the concept of class  $\mathcal{V}(G, h, \varpi)$  (Definition 10). We begin by presenting the concepts and properties of the fuzzy bounded variation functions.

### 3.1 The space of fuzzy bounded variation functions

**Definition 4** (Gong and Wu (2002)) Suppose  $T : a = t_0 < t_1 < \dots < t_n = b$  and  $\tilde{f} : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$  if

$$\sup_T \sum_{i=1}^n ||\tilde{f}(t_i) \ominus_g \tilde{f}(t_{i-1})||$$

is bounded, then  $\tilde{f}$  is fuzzy bounded variation.

For convenience, denote  $V(\tilde{f}, [a, b])$  by the total variation of  $\tilde{f}$ , and  $BV[a, b]$  by the space of fuzzy bounded variation functions.

In order to construct the fuzzy bounded variation function space, we first propose the following:

**Theorem 4**  $||\tilde{f}||$  is of bounded variation if  $\tilde{f} \in BV[a, b]$ .

**Proof** Since  $\tilde{f} \in BV[a, b]$ ,

$$\sup_T \sum_{i=1}^n D(\tilde{f}(t_i), \tilde{f}(t_{i-1})) < \infty.$$

Therefore,

$$\begin{aligned} & |D(\tilde{f}(t_i), \tilde{0}) - D(\tilde{f}(t_{i-1}), \tilde{0})| \\ &= \left| \sup_{\lambda \in [0, 1]} \max\{|f_{\lambda}^{-}(t_i)|, |f_{\lambda}^{+}(t_i)|\} - \sup_{\lambda \in [0, 1]} \max\{|f_{\lambda}^{-}(t_{i-1})|, |f_{\lambda}^{+}(t_{i-1})|\} \right| \\ &= \max\{|f_0^{-}(t_i)|, |f_0^{+}(t_i)|\} - \max\{|f_0^{-}(t_{i-1})|, |f_0^{+}(t_{i-1})|\} \\ &\leq \sup_{\lambda \in [0, 1]} \max\{|f_{\lambda}^{-}(t_i) - f_{\lambda}^{-}(t_{i-1})|, |f_{\lambda}^{+}(t_i) - f_{\lambda}^{+}(t_{i-1})|\} \\ &= D(\tilde{f}(t_i), \tilde{f}(t_{i-1})). \end{aligned}$$

That is

$$\sup_T \sum_{i=1}^n |D(\tilde{f}(t_i), \tilde{0}) - D(\tilde{f}(t_{i-1}), \tilde{0})| < \infty.$$

□

**Corollary 1** If  $\tilde{f}, \tilde{g} \in BV[a, b]$ , then  $D(\tilde{f}(t), \tilde{g}(t))$  is bounded.

Let  $\tilde{f}, \tilde{g} \in BV[a, b]$  and denote

$$D^*(\tilde{f}, \tilde{g}) = \sup_{t \in [a, b]} D(\tilde{f}(t), \tilde{g}(t)),$$

then the mapping  $D^*$  is well-defined by Corollary 1 and we have

**Theorem 5**  $(BV[a, b], D^*)$  is a metric space.

To obtain some completeness, we propose the following theorem:

**Theorem 6** Given the Cauchy sequence  $\{\tilde{f}_n\} \subseteq (BV[a, b], D^*)$  and  $\exists M > 0$  s.t.  $V(\tilde{f}_n, [a, b]) \leq M$  for each  $n$  then  $\tilde{f} \in (BV[a, b], D^*)$  with  $V(\tilde{f}, [a, b]) \leq M$  where  $\tilde{f}$  is the limit of point by point convergence of  $\tilde{f}_n$ .

**Proof** Let  $\{\tilde{f}_n\} \subseteq BV[a, b]$ , and  $V(\tilde{f}_n, [a, b]) \leq M$  for  $\forall n \in \mathcal{N}$ .

Let  $M_n = V(\tilde{f}_n, [a, b])$ . By known conditions, for  $\forall \varepsilon > 0$ ,  $\exists N \in \mathcal{N}$  s.t.

$$\sup_{t \in [a, b]} D(\tilde{f}(t), \tilde{f}_n(t)) < \varepsilon$$

where  $n \geq N$ . For arbitrary partition  $\Gamma$ , let

$$\varepsilon = \frac{1}{N(T) \times m},$$

where  $m > 0$  and  $N(T)$  represent the total number of intervals of  $T$ . Due to

$$D(\tilde{f}(t_i), \tilde{f}(t_{i-1})) \leq D(\tilde{f}(t_i), \tilde{f}_n(t_i)) + D(\tilde{f}_n(t_i), \tilde{f}_n(t_{i-1})) + D(\tilde{f}_n(t_{i-1}), \tilde{f}(t_{i-1})),$$

we have

$$\sup_T \sum_{i=1}^k D(\tilde{f}(t_i), \tilde{f}(t_{i-1})) \leq \frac{2}{m} + \limsup_{n \rightarrow \infty} M_n.$$

From the arbitrariness of  $m$ ,

$$\sup_T \sum_{i=1}^k \|\tilde{f}(t_i) \ominus_g \tilde{f}(t_{i-1})\| \leq M.$$

□

**Definition 5** Let  $M > 0$ , we call  $\tilde{f} : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$  is of fuzzy  $M$ -bounded variation if it is of fuzzy bounded variation with  $V(\tilde{f}, [a, b]) \leq M$ , and write  $\tilde{f} \in MBV[a, b]$ .

**Theorem 7** For  $\forall \tilde{f}, \tilde{g}, \tilde{h} \in MBV[a, b]$ ,

- (i)  $(MBV[a, b], D^*)$  is complete;
- (ii) for  $\forall [a_1, b_1] \subset [a, b]$ ,  $\tilde{f} \in MBV[a_1, b_1]$ ;
- (iii)  $D^*(\tilde{f} + \tilde{g}, \tilde{h} + \tilde{g}) = D^*(\tilde{f}, \tilde{h})$ .

**Definition 6** If there is a finite division  $T$  and some fuzzy numbers  $\tilde{E}_i$  s.t.  $\tilde{f}(t) = \tilde{E}_i$ ,  $t \in (t_{i-1}, t_i)$ , then  $\tilde{f}$  is a finite step fuzzy-valued function.

**Definition 7** If for  $t \in (a, b)$  and  $t \in [a, b)$ , the onesided limits  $\tilde{f}(t-)$ ,  $\tilde{f}(t+)$  exist respectively, then  $\tilde{f}$  is regulated on  $[a, b]$ .

Obviously, the finite step fuzzy-valued function must be fuzzy regulated.

**Theorem 8**  $\tilde{f} \in BV[a, b]$  iff  $f_{\lambda}^{-}, f_{\lambda}^{+}$  are bounded variation functions uniformly about  $\lambda$ .

**Proof** For any division  $T$  of  $[a, b]$ ,

$$\begin{aligned} \sum_{i=1}^n \|\tilde{f}(t_i) \ominus_g \tilde{f}(t_{i-1})\| &= \sum_{i=1}^n \sup_{\lambda \in [0, 1]} \max\{|f_{\lambda}^{-}(t_i) - f_{\lambda}^{-}(t_{i-1})|, |f_{\lambda}^{+}(t_i) - f_{\lambda}^{+}(t_{i-1})|\} \\ &\leq V(\tilde{f}, [a, b]). \end{aligned}$$

Therefore,  $f_{\lambda}^{-}, f_{\lambda}^{+}$  are bounded variation functions uniformly about  $\lambda$  for  $\tilde{f} \in BV[a, b]$ . Conversely,  $\exists M > 0$ , s.t.

$$\sum_{i=1}^n |f_{\lambda}^{-}(t_i) - f_{\lambda}^{-}(t_{i-1})| \leq \frac{M}{2}$$

and

$$\sum_{i=1}^n |f_{\lambda}^{+}(t_i) - f_{\lambda}^{+}(t_{i-1})| \leq \frac{M}{2}$$

for all  $\lambda \in [0, 1]$ , then

$$\sum_{i=1}^n ||\tilde{f}(t_i) \odot_g \tilde{f}(t_{i-1})|| \leq M.$$

□

**Theorem 9** *The right limit of  $\tilde{f}$  at point  $t \in [a, b)$  exists iff there are  $A_{\lambda}^{-}, A_{\lambda}^{+}$  for  $\forall \lambda \in [0, 1]$  s.t.  $f_{\lambda}^{-} \rightarrow A_{\lambda}^{-}$  and  $f_{\lambda}^{+} \rightarrow A_{\lambda}^{+}$  from the right uniformly about for  $\lambda$ .*

**Proof** There is  $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$  s.t.  $\lim_{\delta \rightarrow 0^{+}} \tilde{f}(t + \delta) = \tilde{A}$ , then

$$\limsup_{\delta \rightarrow 0^{+}, \lambda \in [0, 1]} \max\{|f_{\lambda}^{-}(t + \delta) - A_{\lambda}^{-}|, |f_{\lambda}^{+}(t + \delta) - A_{\lambda}^{+}|\} = 0.$$

Conversely, for  $\forall \lambda \in [0, 1]$  there are  $A_{\lambda}^{-}, A_{\lambda}^{+}$  s.t.  $f_{\lambda}^{-} \rightarrow A_{\lambda}^{-}$  and  $f_{\lambda}^{+} \rightarrow A_{\lambda}^{+}$  uniformly about for  $\lambda$ , then by the properties of fuzzy numbers, there is  $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$  s.t.  $[\tilde{A}]_{\lambda} = [A_{\lambda}^{-}, A_{\lambda}^{+}]$  for all  $\lambda \in [0, 1]$ . □

**Corollary 2** *If  $\tilde{f} \in BV[a, b]$ , then it's fuzzy regulated.*

From Theorem 8 and 9, it can be obtained easily.

**Theorem 10** *For any fuzzy bounded variation function  $\tilde{f}$ , there is a sequence of finite step fuzzy-number-valued functions  $\{\tilde{f}_n\}$  s.t.  $\tilde{f}_n \rightarrow \tilde{f}$ .*

**Proof** Let  $\tilde{F}(t) \triangleq V(\tilde{f}, [a, t])$  and  $\tilde{f}(t) = \tilde{f}(a)$ . It's obvious that  $\tilde{F}(t)$  is bounded and nondecreasing with  $\tilde{F}(a) = 0, \tilde{F}(b) = V(\tilde{f}, [a, b])$ .

Let  $a = c_0, b = c_{n+1}$  and  $c_i \in [a, b], i = 0, \dots, n$  s.t.

$$\tilde{F}(c_i^{-}) \leq \frac{i}{n} V(\tilde{f}, [a, b]),$$

and

$$\tilde{F}(c_i^{+}) \geq \frac{i}{n} V(\tilde{f}, [a, b]),$$

where  $c_i$  exists by Corollary 2.

Then we let

$$\tilde{f}_{n+1}(t) = \begin{cases} \tilde{f}(c_i), & t = c_i, i = 0, \dots, n+1, \\ \tilde{f}(c_i^{+}), & t \in (c_i, c_{i+1}), i = 0, \dots, n. \end{cases}$$

From the construction of  $\tilde{f}_n$ ,

$$D^{*}(\tilde{f}_n, \tilde{f}) \leq \frac{1}{n} V(\tilde{f}, [a, b]).$$

□

**Theorem 11**  *$MBV[a, b] \cap C([a, b], \mathbb{R}_{\mathcal{F}})$  is closed and convex.*

**Proof** From (i) of Theorem 7, we derive that  $MBV[a, b]$  is closed in  $C([a, b], \mathbb{R}_{\mathcal{F}})$ . Suppose  $\tilde{f}, \tilde{g} \in MBV[a, b]$  and  $\beta \in [0, 1]$ , then for any division, we have

$$\begin{aligned} & \sum_{i=1}^n \|(\beta \tilde{f} + (1 - \beta) \tilde{g})(t_i) \ominus_g (\beta \tilde{f} + (1 - \beta) \tilde{g})(t_{i-1})\| \\ &= \sum_{i=1}^n D((\beta \tilde{f} + (1 - \beta) \tilde{g})(t_i), (\beta \tilde{f} + (1 - \beta) \tilde{g})(t_{i-1})) \\ &\leq \sum_{i=1}^n \left( D(\beta \tilde{f}(t_i), \beta \tilde{f}(t_{i-1})) + D((1 - \beta) \tilde{g}(t_i), (1 - \beta) \tilde{g}(t_{i-1})) \right) \\ &\leq \beta \sum_{i=1}^n D(\tilde{f}(t_i), \tilde{f}(t_{i-1})) + (1 - \beta) \sum_{i=1}^n D(\tilde{g}(t_i), \tilde{g}(t_{i-1})) \\ &\leq \beta V(\tilde{f}, [a, b]) + (1 - \beta) V(\tilde{g}, [a, b]) \\ &\leq \beta M + (1 - \beta) M = M. \end{aligned}$$

That is  $\beta \tilde{f} + (1 - \beta) \tilde{g} \in MBV[a, b]$ .  $\square$

### 3.2 The BV-solutions for GFDEs

In this subsection, we consider the following GFDEs

$$\tilde{x}'_g(t) = \tilde{f}(t, \tilde{x}(t)), \quad \tilde{x}(t_0) = \tilde{x}_0 \quad (2)$$

where  $\tilde{x}_0 \in \mathbb{R}_{\mathcal{F}}$  and  $\tilde{f} : G \rightarrow \mathbb{R}_{\mathcal{F}}$  is fuzzy Henstock integrable. Denote by  $G \triangleq I \times B_c$  where  $I = [a, b]$  and  $B_c = \{\tilde{u} \in \mathbb{R}_{\mathcal{F}}, \|\tilde{u}\| < c\}$ .

Next, we will provide the definition of BV-solutions for problem (2) based on the fuzzy Henstock integral.

**Definition 8** We call  $\tilde{x}(t)$  a BV-solutions of (2) on  $[\alpha, \beta] \subset \mathbb{R}$  iff it is continuous and of fuzzy bounded variation on  $[\alpha, \beta]$ , and satisfies the following integral equations

$$\begin{cases} \tilde{x}(t) \ominus_g \tilde{x}_0 = \int_{t_0}^t \tilde{f}(s, \tilde{x}(s)) ds \\ \tilde{x}(t_0) = \tilde{x}_0 \end{cases}$$

**Definition 9** (Xue and Fu (2002)) Let  $\tilde{f} : G \rightarrow \mathbb{R}_{\mathcal{F}}$ , if

- (i)  $\tilde{f}(t, \cdot)$  is continuous a.e.  $t \in [a, b]$ ;
- (ii)  $\tilde{f}(\cdot, \tilde{x})$  is measurable for  $\forall \tilde{x} \in B_c$ .

then  $\tilde{f}$  is called the Carathéodory function.

**Definition 10** A Carathéodory function  $\tilde{f} : G \rightarrow \mathbb{R}_{\mathcal{F}}$  belong to  $\mathcal{V}(G, h, \varpi)$  if

- (a) there is a  $\delta : I \rightarrow (0, \infty)$ , for  $\forall \tau \in [u, v] \subset (\tau - \delta(\tau), \tau + \delta(\tau)) \subset I$  and  $\tilde{x} \in B_c$

$$D(\tilde{f}(\tau, \tilde{x})(v - u), \tilde{0}) \leq h(v) - h(u);$$

- (b) for  $\forall \tilde{x}, \tilde{y} \in B_c$  and  $\tau \in [u, v] \subset (\tau - \delta(\tau), \tau + \delta(\tau)) \subset I$

$$\|\tilde{f}(\tau, \tilde{x}) \ominus_g \tilde{f}(\tau, \tilde{y})\|(v - u) \leq \varpi(\|\tilde{x} \ominus_g \tilde{y}\|)(h(v) - h(u));$$

(c) for each finite step fuzzy-valued function  $\tilde{x}(t)$ ,  $\tilde{f}(t, \tilde{x})$  is fuzzy Henstock integrable,

where  $h : I \rightarrow (-\infty, \infty)$  is increasing and continuous from the left,  $\varpi \in \mathcal{K}$  i.e.  $\varpi : [0, \infty) \rightarrow [0, \infty)$  is increasing with continuous and satisfies  $\varpi(0) = 0$ .

**Theorem 12** Let  $\tilde{f} : G \rightarrow \mathbb{R}_{\mathcal{F}} \in \mathcal{V}(G, h, \varpi)$  and  $[\alpha, \beta] \subset [a, b]$ . If  $\tilde{x} : [\alpha, \beta] \rightarrow \mathbb{R}_{\mathcal{F}}$  is a BV-solution of (2), then  $\tilde{x} \in BV[\alpha, \beta]$  with

$$V(\tilde{x}, [\alpha, \beta]) \leq h(\beta) - h(\alpha) < \infty.$$

**Proof** For any  $\Gamma$  of  $[\alpha, \beta]$ .  $\int_{t_{j-1}}^{t_j} \tilde{f}(t, \tilde{x}(t))dt$  exists on  $[t_{j-1}, t_j]$  for all  $j = 1, \dots, n$ , then for  $\forall \varepsilon > 0$  there is a function  $\delta_i : [t_{j-1}, t_j] \rightarrow (0, \infty)$  s.t. for each  $\delta_i$ -fine partition:  $\Gamma_i$ ,  $\xi_i^j \in [t_{i-1}^j, t_i^j] \subset (\xi_i^j - \delta_j(\xi_i^j), \xi_i^j + \delta_j(\xi_i^j))$  for all  $i = 1, 2, \dots, m_k$ . From (a)

$$\begin{aligned} \left\| \int_{t_{j-1}}^{t_j} \tilde{f}(t, \tilde{x}(t))dt \right\| &\leq \left\| \int_{t_{j-1}}^{t_j} \tilde{f}(t, \tilde{x}(t))dt \odot_g \vartheta(\tilde{f}(t, \tilde{x}), \Gamma_i) \right\| + \|\vartheta(\tilde{f}(t, \tilde{x}), \Gamma_i)\| \\ &\leq \frac{\varepsilon}{2^j} + h(t_j) - h(t_{j-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j=1}^k \|\tilde{x}(t_j) \odot_g \tilde{x}(t_{j-1})\| &= \sum_{j=1}^k \left\| \int_{t_{j-1}}^{t_j} \tilde{f}(t, \tilde{x}(t))dt \right\| \\ &< \varepsilon + \sum_{j=1}^k (h(t_j) - h(t_{j-1})) = \varepsilon + h(\beta) - h(\alpha). \end{aligned}$$

□

**Theorem 13** Suppose a fuzzy sequence  $\tilde{x}_n \rightarrow \tilde{x}$  and  $\tilde{f} : G \rightarrow \mathbb{R}_{\mathcal{F}} \in \mathcal{V}(G, h, \varpi)$ . For  $\forall s \in [\alpha, \beta]$ ,  $(s, \tilde{x}(s)), (s, \tilde{x}_n(s)) \in G$  and  $\int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}_n(t))dt$  exists, then  $\int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt$  exists with

$$\lim_{n \rightarrow \infty} \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}_n(t))dt = \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt.$$

**Proof** By Theorem 2, for  $\forall \varepsilon > 0$ , since  $\tilde{f} \in FH[\alpha, \beta]$ ,  $\exists \delta_n(\xi) > 0$  s.t.

$$D\left(\vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma_1), \vartheta(\tilde{f}(t, \tilde{x}), \Gamma_2)\right) < \varepsilon,$$

for every  $\Gamma_1$  and  $\Gamma_2$  which are  $\delta_n$ -fine.

Let  $\delta(\xi) \triangleq \delta_n(\xi)$ , then the  $\delta$ -fine partition  $\Gamma$  is also  $\delta_n$ -fine. From  $D(\tilde{f}(\tau, \tilde{x}), \tilde{f}(\tau, \tilde{y}))(v - u) \leq \varpi(D(\tilde{x}, \tilde{y}))(h(v) - h(u))$ ,

$$\begin{aligned} D\left(\vartheta(\tilde{f}(t, \tilde{x}), \Gamma), \vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma)\right) &\leq \sum D\left(\tilde{f}(\xi, \tilde{x}(\xi)), \tilde{f}(\xi, \tilde{x}_n(\xi))\right)(v - u) \\ &\leq \sum \varpi(D(\tilde{x}(\xi), \tilde{x}_n(\xi)))(h(v) - h(u)) \\ &\leq \sup_{t \in [\alpha, \beta]} \varpi(D(\tilde{x}(t), \tilde{x}_n(t)))(h(\beta) - h(\alpha)). \end{aligned}$$

Due to  $\tilde{x}_n \rightarrow \tilde{x}$  and  $\varpi \in \mathcal{K}$ ,

$$D\left(\vartheta(\tilde{f}(t, \tilde{x}), \Gamma), \vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma)\right) \rightarrow 0.$$

For any  $\Gamma_1$  and  $\Gamma_2$  which are  $\delta$ -fine,

$$\begin{aligned} & D\left(\vartheta(\tilde{f}(t, \tilde{x}), \Gamma_1), \vartheta(\tilde{f}(t, \tilde{x}), \Gamma_2)\right) \\ & \leq D\left(\vartheta(\tilde{f}(t, \tilde{x}), \Gamma_1), \vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma_1)\right) + D\left(\vartheta(\tilde{f}(t, \tilde{x}), \Gamma_2), \vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma_2)\right) \\ & + D\left(\vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma_1), \vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma_2)\right) \leq \varepsilon. \end{aligned}$$

By Theorem 2,  $\int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt$  exists. Also,

$$\begin{aligned} & D\left(\int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}_n(t))dt, \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt\right) \\ & \leq D\left(\vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma), \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}_n(t))dt\right) + D\left(\vartheta(\tilde{f}(t, \tilde{x}), \Gamma), \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt\right) \\ & + D\left(\vartheta(\tilde{f}(t, \tilde{x}), \Gamma), \vartheta(\tilde{f}(t, \tilde{x}_n), \Gamma)\right) \leq \varepsilon. \end{aligned}$$

So,  $\int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}_n(t))dt \rightarrow \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt$ .  $\square$

**Corollary 3** Suppose a sequence of finite step fuzzy-number-valued functions  $\tilde{x}_n \rightarrow \tilde{x}$  and  $\tilde{f} : G \rightarrow \mathbb{R}_{\mathcal{F}} \in \mathcal{V}(G, \tilde{h}, \varpi)$ . For  $\forall s \in [\alpha, \beta]$ ,  $(s, \tilde{x}(s)), (s, \tilde{x}_n(s)) \in G$  and  $\int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}_n(t))dt$  exists, then  $\int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt$  exists with

$$\lim_{n \rightarrow \infty} \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}_n(t))dt = \int_{\alpha}^{\beta} \tilde{f}(t, \tilde{x}(t))dt.$$

**Corollary 4** If  $\tilde{f} \in \mathcal{V}(G, \tilde{h}, \varpi)$  and  $\tilde{x} \in BV[\alpha, \beta]$ , then by Corollary 3 and Theorem 10  $\int_{\alpha}^{\beta} \tilde{f}(s, \tilde{x}(s))ds$  exists.

## 4 Existence of BV-solutions for GFDEs

In this section, for the fuzzy Henstock integrable function  $\tilde{f} \in \mathcal{V}(G, \tilde{h}, \varpi)$  of problem (2), we obtain some existence theorems.

Next, the local existence theorems of BV-solutions for problem (2) will be established where  $\tilde{f} \in \mathcal{V}(G, \tilde{h}, \varpi)$  with  $\tilde{x}(t_0) = \tilde{x}_0 \in \mathbb{R}_{\mathcal{F}}$ .

**Theorem 14** Let  $(t_0, \tilde{x}_0) \in G$  and  $\tilde{f} \in \mathcal{V}(G, \tilde{h}, \varpi)$ . Then there exist  $\Delta^-, \Delta^+ > 0$  s.t. on  $[t_0 - \Delta^-, t_0 + \Delta^+]$  there is a BV-solution  $\tilde{x}$  of (2) with  $\tilde{x}(t_0) = \tilde{x}_0$ .

**Proof** Since  $B_c$  is a open ball in  $\mathbb{R}_{\mathcal{F}}$  there is a  $\Delta^- > 0$  s.t. if  $t \in [t_0 - \Delta^-, t_0]$  and  $\tilde{x} \in \mathbb{R}_{\mathcal{F}}$  is s.t.  $\|\tilde{x} \ominus_g \tilde{x}_0\| < |\tilde{h}(t) - \tilde{h}(t_0)|$  then  $(\tilde{x}, t) \in G = I \times B_c$ , and similarly there exists a  $\Delta^+ > 0$  s.t. if  $t \in [t_0, t_0 + \Delta^+]$  and  $\tilde{x} \in \mathbb{R}_{\mathcal{F}}$  is s.t.  $\|\tilde{x} \ominus_g \tilde{x}_0\| < |\tilde{h}(t) - \tilde{h}(t_0^+)|$ , then  $(t, \tilde{x}) \in G$  where  $\tilde{h}(t_0^+) = \lim_{t \rightarrow t_0^+} \tilde{h}(t)$ .

Denote  $\mathcal{A}$  by the set of all functions  $\tilde{q} : [t_0 - \Delta^-, t_0 + \Delta^+] \rightarrow \mathbb{R}_{\mathcal{F}}$  s.t.

$$\tilde{q} \in MBV[t_0 - \Delta^-, t_0 + \Delta^+] \cap C[t_0 - \Delta^-, t_0 + \Delta^+],$$

$$||\tilde{q}(t) \odot_g \tilde{x}_0|| < |\tilde{h}(t) - \tilde{h}(t_0)|$$

for  $t \in [t_0 - \Delta^-, t_0]$ ,

$$||\tilde{q}(t) \odot_g \tilde{x}_0|| < |\tilde{h}(t) - \tilde{h}(t_0^+)|$$

for  $t \in (t_0, t_0 + \Delta^+]$ , where  $M = \tilde{h}(b) - \tilde{h}(a)$ .

Obviously,  $\mathcal{A} \subset MBV[t_0 - \Delta^-, t_0 + \Delta^+] \cap C[t_0 - \Delta^-, t_0 + \Delta^+]$  is a convex subset, i.e. if  $\tilde{q}_1, \tilde{q}_2 \in \mathcal{A}$ ,  $\alpha \in [0, 1]$  then  $\alpha\tilde{q}_1 + (1 - \alpha)\tilde{q}_2 \in \mathcal{A}$ .

From the definition of  $\mathcal{A}$ ,  $\tilde{q}(t_0) = \tilde{x}_0$  for  $\tilde{q} \in \mathcal{A}$ . Next, let us show  $\mathcal{A}$  is closed. Let  $\{\tilde{q}_k\} \subset \mathcal{A}$  be a sequence which converges in  $MBV[t_0 - \Delta^-, t_0 + \Delta^+] \cap C[t_0 - \Delta^-, t_0 + \Delta^+]$  to a function  $\tilde{q}$ . Let  $|| \cdot ||_{\mathcal{X}} \triangleq || \cdot ||_{\mathcal{X}[t_0 - \Delta^-, t_0 + \Delta^+]}$ , since

$$||\tilde{q}_k(t) \odot_g \tilde{q}(t)|| \leq ||\tilde{q}_k \odot_g \tilde{q}||_{\mathcal{X}}$$

for  $t \in [t_0 - \Delta^-, t_0 + \Delta^+]$ ,

$$\lim_{k \rightarrow \infty} ||\tilde{q}_k(t) \odot_g \tilde{q}(t)|| = 0$$

uniformly for  $t \in [t_0 - \Delta^-, t_0 + \Delta^+]$ . Therefore, for  $\forall \varepsilon > 0, \exists k \in \mathcal{N}$  s.t. for  $t \in (t_0, t_0 + \Delta^+]$ ,

$$\begin{aligned} ||\tilde{q}(t) \odot_g \tilde{x}_0|| &\leq ||\tilde{q}(t) \odot_g \tilde{q}_k(t)|| + ||\tilde{q}_k(t) \odot_g \tilde{x}_0|| \\ &< \varepsilon + |\tilde{h}(t) - \tilde{h}(t_0^+)|. \end{aligned}$$

This means for  $t \in (t_0, t_0 + \Delta^+]$

$$||\tilde{q}(t) \odot_g \tilde{x}_0|| \leq |\tilde{h}(t) - \tilde{h}(t_0^+)|.$$

Similarly, for  $t \in [t_0 - \Delta^-, t_0]$

$$||\tilde{q}(t) \odot_g \tilde{x}_0|| \leq |\tilde{h}(t) - \tilde{h}(t_0)|$$

and therefore  $\tilde{q} \in \mathcal{A}$  and  $\mathcal{A}$  is closed. From Theorem 3 and Theorem 11,  $\mathcal{A}$  can be regarded as a closed convex subset in  $\mathcal{X}[t_0 - \Delta^-, t_0 + \Delta^+]$ .

For a given  $\tilde{q} \in \mathcal{A}$  define the mapping

$$T\tilde{q}(t) \odot_g \tilde{x}_0 = \int_{t_0}^t \tilde{f}(s, \tilde{q}(s))ds, \quad t \in [t_0 - \Delta^-, t_0 + \Delta^+].$$

By Corollary 4,  $T$  is well-defined. Due to  $\int_{t_0}^t \tilde{f}(s, \tilde{q}(s))ds$  exists for  $\forall t \in [t_0 - \Delta^-, t_0 + \Delta^+]$  where  $\tilde{q} \in \mathcal{A}$ , then  $\int_c^t \tilde{f}(s, \tilde{q}(s))ds$  exists for every  $[c, t] \subset (t_0, t_0 + \Delta^+]$ ,  $\tilde{q} \in \mathcal{A}$ .

For  $\forall \varepsilon > 0, \exists \delta(\xi) : [c, t] \rightarrow (0, \infty)$  s.t. for  $\Gamma = \{[c_{i-1}, c_i], \xi_i\}, i = 1, 2, \dots, m$  which is  $\delta$ -fine.

$$\begin{aligned} ||T\tilde{q}(t) \odot_g \tilde{x}_0|| &= ||\int_{t_0}^t \tilde{f}(s, \tilde{q}(s))ds|| = \lim_{\sigma \rightarrow t_0^+} ||\int_{\sigma}^t \tilde{f}(s, \tilde{q}(s))ds|| \\ &\leq \lim_{\sigma \rightarrow t_0^+} \left[ ||\int_{\sigma}^t \tilde{f}(s, \tilde{q}(s))ds \odot_g \sum_{i=1}^m \tilde{f}(\xi_i, \tilde{q}(\xi_i))(c_i - c_{i-1})|| \right. \\ &\quad \left. + ||\sum_{i=1}^m \tilde{f}(\xi_i, \tilde{q}(\xi_i))(c_i - c_{i-1})|| \right] \\ &< \varepsilon + \lim_{\sigma \rightarrow t_0^+} \sum_{i=1}^m |\tilde{h}(c_i) - \tilde{h}(c_{i-1})| = \varepsilon + \lim_{\sigma \rightarrow t_0^+} (\tilde{h}(t) - \tilde{h}(c)) \\ &= \varepsilon + |\tilde{h}(t) - \tilde{h}(t_0^+)|. \end{aligned}$$

This means for  $t \in (t_0, t_0 + \Delta^+]$

$$\|T\tilde{q}(t) \ominus_g \tilde{x}_0\| \leq |\tilde{h}(t) - \tilde{h}(t_0^+)|.$$

Similarly, for  $t \in [t_0 - \Delta^-, t_0]$ ,

$$\|T\tilde{q}(t) \ominus_g \tilde{x}_0\| \leq |\tilde{h}(t) - \tilde{h}(t_0)|.$$

Therefore,  $T\tilde{q} \in \mathcal{A}$  for  $\tilde{q} \in \mathcal{A}$ , i.e.  $T : \mathcal{A} \rightarrow \mathcal{A}$ . Next, we shall show  $T : \mathcal{A} \rightarrow \mathcal{A}$  is continuous. If  $\tilde{q}, \tilde{l} \in \mathcal{A}$ , then

$$\begin{aligned} \|T\tilde{q} \ominus_g T\tilde{l}\|_{\mathcal{X}} &= \sup_{t \in [t_0 - \Delta^-, t_0 + \Delta^+]} \|T\tilde{q}(t) \ominus_g T\tilde{l}(t)\| \\ &\leq \sup_{t \in [t_0 - \Delta^-, t_0 + \Delta^+]} \left\| \int_{t_0}^t \tilde{f}(s, \tilde{q}(s)) \ominus_g \tilde{f}(s, \tilde{l}(s)) ds \right\|. \end{aligned}$$

Let  $\Gamma : t_0 - \Delta^- = u_0 < u_1 < \dots < u_k = t_0 + \Delta^+$  be any partition of  $[t_0 - \Delta^-, t_0 + \Delta^+]$ . Due to  $\int_{u_{j-1}}^{u_j} \tilde{f}(s, \tilde{q}(s)) \ominus_g \tilde{f}(s, \tilde{l}(s)) ds$  exists on  $[u_{j-1}, u_j]$  for all  $j = 1, 2, \dots, k$ . For  $\forall \varepsilon > 0$  there is  $\delta_j : [u_{j-1}, u_j] \rightarrow (0, \infty)$  s.t. for any  $\delta_j$ -fine partition  $\Gamma^* : u_{j-1} = t_0^j < t_1^j < \dots < t_{m_j}^j = u_j$  and  $\{\theta_1^j, \theta_2^j, \dots, \theta_{m_j}^j\}$  satisfied that  $\theta_i^j \in [t_{i-1}^j, t_i^j] \subset (\theta_i^j - \delta(\theta_i^j), \theta_i^j + \delta(\theta_i^j))$ , we have

$$\begin{aligned} &\sup_{t \in [t_0 - \Delta^-, t_0 + \Delta^+]} \left\| \int_{t_0}^t \tilde{f}(s, \tilde{q}(s)) \ominus_g \tilde{f}(s, \tilde{l}(s)) ds \right\| \\ &\leq \sup_{t \in [t_0 - \Delta^-, t_0 + \Delta^+]} \left[ \left\| \left( \int_{t_0}^t \tilde{f}(s, \tilde{q}(s)) \ominus_g \tilde{f}(s, \tilde{l}(s)) ds \right) \right\| \right. \\ &\quad \ominus_g \left( \sum_{i=1}^m (\tilde{f}(\xi_i, \tilde{q}(\xi_i)) \ominus_g \tilde{f}(\xi_i, \tilde{l}(\xi_i)))(c_i - c_{i-1}) \right) \Big\| \\ &\quad \left. + \left\| \sum_{i=1}^m (\tilde{f}(\xi_i, \tilde{q}(\xi_i)) \ominus_g \tilde{f}(\xi_i, \tilde{l}(\xi_i)))(c_i - c_{i-1}) \right\| \right] \\ &\leq \sup_{t \in [t_0 - \Delta^-, t_0 + \Delta^+]} \left[ \varepsilon + \sum_{i=1}^m \varpi(\|\tilde{q}(\xi_i) \ominus_g \tilde{l}(\xi_i)\|)(\tilde{h}(c_i) - \tilde{h}(c_{i-1})) \right] \\ &\leq \varepsilon + \int_{t_0 - \Delta^-}^{t_0 + \Delta^+} \varpi(\|\tilde{q}(t) \ominus_g \tilde{l}(t)\|) d\tilde{h}(t), \end{aligned}$$

that is

$$\|T\tilde{q} \ominus_g T\tilde{l}\|_{\mathcal{X}} < \varepsilon + \int_{t_0 - \Delta^-}^{t_0 + \Delta^+} \varpi(\|\tilde{q}(t) \ominus_g \tilde{l}(t)\|) d\tilde{h}(t).$$

Suppose  $\tilde{q}, \tilde{q}_k \in \mathcal{A}$  with  $\lim_{k \rightarrow \infty} \|\tilde{q}_k \ominus_g \tilde{q}\|_{\mathcal{X}} = 0$ . Then by

$$\lim_{k \rightarrow \infty} \|\tilde{q}_k(t) \ominus_g \tilde{q}(t)\| = 0$$

uniformly for  $t \in [t_0 - \Delta^-, t_0 + \Delta^+]$  and since  $\varpi$  is continuous at 0 with  $\varpi(0) = 0$ ,

$$\lim_{k \rightarrow \infty} \|T\tilde{q}_k \ominus_g T\tilde{q}\|_{\mathcal{X}} = 0.$$

i.e.  $T$  is continuous.

Finally we shall prove  $T : \mathcal{A} \rightarrow \mathcal{A}$  is compact. Let  $\mathbb{K} \in \mathcal{A}$  s.t.  $\exists N > 0$  s.t.  $\|\tilde{q}\|_{\mathcal{X}} \leq N$  for  $\tilde{q} \in \mathbb{K}$ .

Therefore the sequence of equally bounded functions  $\{\tilde{q}_k\}$  is of equibounded variation where  $\tilde{q}_k \in \mathbb{K}, k \in \mathcal{N}$ . That means this sequence contains a pointwise convergent subsequence which we again denote by  $\{\tilde{q}_k\}$  by Helly's Choice Theorem. Therefore, for  $t \in [t_0 - \Delta^-, t_0 + \Delta^+]$

$$\lim_{k \rightarrow \infty} \tilde{q}_k(t) = \tilde{q}(t)$$

where  $\tilde{q} \in MBV[t_0 - \Delta^-, t_0 + \Delta^+] \cap C[t_0 - \Delta^-, t_0 + \Delta^+]$ .

Put

$$\tilde{y}(t) \odot_g \tilde{x}_0 = T\tilde{q}(t) \odot_g \tilde{x}_0 = \int_{t_0}^t \tilde{f}(s, \tilde{q}(s))ds$$

for  $s \in [t_0 - \Delta^-, t_0 + \Delta^+]$ . It can be obtained from the above certificate that

$$\lim_{k \rightarrow \infty} \|T\tilde{q}_k \odot_g \tilde{y}\|_{\mathcal{X}} = 0.$$

Therefore, we have  $T : \mathcal{A} \rightarrow \mathcal{A}$  is compact.

By Schauder-Tichonov's fixed point theorem, there is a  $\tilde{x} \in \mathcal{A}$  s.t.  $\tilde{x} = T\tilde{x}$ , i.e.

$$\tilde{x}(t) \odot_g \tilde{x}_0 = T(\tilde{x}(t)) \odot_g \tilde{x}_0 = \int_{t_0}^t \tilde{f}(s, \tilde{x}(s))ds$$

for  $t \in [t_0 - \Delta^-, t_0 + \Delta^+]$  and

$$T\tilde{x}(t_2) \odot_g T\tilde{x}(t_1) = \int_{t_1}^{t_2} \tilde{f}(s, \tilde{x}(s))ds$$

and this implies that  $\tilde{x}$  is a BV-solution of (2).

Next, we will provide a simple example to prove the existence of bounded variation solutions for linear fuzzy differential equations using our method.  $\square$

**Example 3** Considering problem (2), let  $c > 0, I = [a, b], \tilde{x}_0 \in \mathbb{R}_{\mathcal{F}}$  with  $[\tilde{x}_0]_{\lambda} = [\lambda, \lambda + (1 - \lambda)\frac{c}{2}]$  and

$$\tilde{f}(t, \tilde{x}) = t\tilde{A} + \tilde{x},$$

where  $\tilde{A} \in \mathbb{R}_{\mathcal{F}}$ .

Let

$$h(t) = (c + b\|\tilde{A}\|)t$$

and

$$\varpi(t) = \frac{t}{c + b\|\tilde{A}\|}.$$

Obviously,  $h : I \rightarrow (-\infty, \infty)$  is increasing and continuous from the left,  $\varpi \in \mathcal{K}$ , and  $\tilde{f}$  is a Carathéodory function. Next, let us show  $\tilde{f} \in \mathcal{V}(G, h, \varpi)$ .

(a) there is a  $\delta(t) = \frac{1}{t}$ , for  $\forall \tau \in [u, v] \subset (\tau - \frac{1}{t}, \tau + \frac{1}{t}) \subset I$  and  $\tilde{x} \in B_c$

$$D(\tilde{f}(\tau, \tilde{x})(v - u), \tilde{0}) \leq (\|\tilde{x}\| + \tau\|\tilde{A}\|)(v - u) \leq (c + b\|\tilde{A}\|)(v - u) \leq h(v) - h(u);$$

(b) for  $\forall \tilde{x}, \tilde{y} \in B_c, \tau \in [u, v] \subset (\tau - \frac{1}{t}, \tau + \frac{1}{t}) \subset I$

$$\begin{aligned} \|\tilde{f}(\tau, \tilde{x}) \odot_g \tilde{f}(\tau, \tilde{y})\|(v - u) &\leq \|\tilde{x} \odot_g \tilde{y}\|(v - u) \leq \frac{\|\tilde{x} \odot_g \tilde{y}\|}{(c + b\|\tilde{A}\|)}(c + b\|\tilde{A}\|)(v - u) \\ &\leq \varpi(\|\tilde{x} \odot_g \tilde{y}\|)(h(v) - h(u)) \end{aligned}$$

(c) Obviously, for each finite step fuzzy-valued function  $\tilde{x}(t)$ ,  $\tilde{f}(t, \tilde{x})$  is fuzzy Henstock integrable.

From Theorem 14, there exist  $\Delta^-, \Delta^+ > 0$  s.t. on  $[t_0 - \Delta^-, t_0 + \Delta^+]$  there is a BV-solution  $\tilde{x}$  of (2) with  $\tilde{x}(t_0) = \tilde{x}_0$ .

For slightly more complex fuzzy differential equations, our method can also be used to prove the existence of bounded variation solutions.

**Example 4** Now, we consider the GFDEs as follows,  $c > 0$ ,  $I = [a, b]$ ,  $\tilde{x}_0 \in \mathbb{R}_{\mathcal{F}}$  with  $[\tilde{x}_0]_{\lambda} = [\frac{c}{2}(\lambda - 1), \frac{c}{2}(1 - \lambda)]$  and

$$\tilde{f}(t, \tilde{x}) = \frac{e^t}{t} \tilde{x}.$$

Let

$$h(t) = c \cdot e^t \quad \text{and} \quad \varpi(t) = \frac{t}{c}.$$

Then,  $h : I \rightarrow (-\infty, \infty)$  is increasing and continuous from the left,  $\varpi \in \mathcal{K}$ , and  $\tilde{f}$  is a Carathéodory function. Similarly, we will show  $\tilde{f} \in \mathcal{V}(G, h, \varpi)$ .

(a) there is a  $\delta(t) = \frac{t}{2}$ , for  $\forall \tau \in [u, v] \subset (\frac{\tau}{2}, \frac{3\tau}{2}) \subset I$  and  $\tilde{x} \in B_c$

$$\begin{aligned} D(\tilde{f}(\tau, \tilde{x})(v - u), \tilde{0}) &\leq \frac{e^{\tau}}{\tau} \|\tilde{x}\| (v - u) \leq c \cdot \frac{e^{\tau}}{\tau} (v - u) \leq c \cdot \left( \frac{e^v}{v} - \frac{e^u}{u} \right) \\ &= c \cdot (e^v - e^u) = h(v) - h(u); \end{aligned}$$

(b) for  $\forall \tilde{x}, \tilde{y} \in B_c$ ,  $\tau \in [u, v] \subset (\frac{\tau}{2}, \frac{3\tau}{2}) \subset I$

$$\begin{aligned} \|\tilde{f}(\tau, \tilde{x}) \ominus_g \tilde{f}(\tau, \tilde{y})\| (v - u) &\leq \frac{e^{\tau}}{\tau} \|\tilde{x} \ominus_g \tilde{y}\| (v - u) \leq \frac{\|\tilde{x} \ominus_g \tilde{y}\|}{c} \cdot c \cdot (e^v - e^u) \\ &\leq \varpi(\|\tilde{x} \ominus_g \tilde{y}\|) (h(v) - h(u)) \end{aligned}$$

(c) Obviously, for each finite step fuzzy-valued function  $\tilde{x}(t)$ ,  $\tilde{f}(t, \tilde{x})$  is fuzzy Henstock integrable.

Therefore, by Theorem 14, there is a BV-solution  $\tilde{x}$  of (2) with  $\tilde{x}(t_0) = \tilde{x}_0$ .

To demonstrate that this paper extends the existing work, we will provide an example to illustrate the existence of a fuzzy differential equation with bounded variation solutions in the sense of g-differential, but there are no (i) and (ii) solutions in the sense of gH-differential.

**Example 5** Let us considering problem (2) given in Example 1. Let

$$h(t) = (V(g, [a, b]) + g(a) + 2)t$$

and

$$\varpi(t) = t,$$

where  $V(g, [a, b])$  represent the total variation of real-valued function  $g$ . Obviously,  $h : I \rightarrow (-\infty, \infty)$  is increasing and continuous from the left,  $\varpi \in \mathcal{K}$ , and  $\tilde{f}$  is a Carathéodory function.

First, we prove that the fuzzy differential equation does not have a solution in the sense of gH-differential. For the solution  $\tilde{x}(t) = \chi_{G(t)} + \tilde{A}t$ , where

$$G(t) = \begin{cases} t^2 \sin \frac{1}{t^2}, & t \neq 0 \\ 0, & t = 0, \end{cases}$$

we can easily find that the diameters of solution equals to 0 for  $t = 0$ , and greater than 0 for  $t \neq 0$ . That means there no (i) and (ii) solutions in the sense of gH-differential for  $a < 0 < b$ . Next, let us show  $\tilde{f} \in \mathcal{V}(G, \tilde{h}, \varpi)$ .

(a) there is a  $\delta(t) = \frac{t}{2}$ , for  $\forall \tau \in [u, v] \subset (\tau - \frac{\tau}{2}, \tau + \frac{\tau}{2}) \subset I$  and  $\tilde{x} \in B_c$

$$\begin{aligned} D(\tilde{f}(\tau, \tilde{x})(v - u), \tilde{0}) &= D((\chi_{g(t)} + \tilde{A})(v - u), \tilde{0}) \\ &\leq D(\chi_{g(t)}(v - u), \tilde{0}) + D(\tilde{A}(v - u), \tilde{0}) \\ &\leq \max_{\tau \in [u, v]} \left\{ 2\tau \sin \frac{1}{\tau^2} - \frac{2}{\tau} \cos \frac{1}{\tau^2} \right\} (v - u) + 2(v - u) \\ &\leq (V(g, [u, v]) + g(u) + 2)(v - u) \\ &\leq (V(g, [a, b]) + g(a) + 2)(v - u) \\ &\leq \tilde{h}(v) - \tilde{h}(u); \end{aligned}$$

(b) for  $\forall \tilde{x}, \tilde{y} \in B_c, \tau \in [u, v] \subset (\tau - \frac{\tau}{2}, \tau + \frac{\tau}{2}) \subset I$

$$\|\tilde{f}(\tau, \tilde{x}) \ominus_g \tilde{f}(\tau, \tilde{y})\|(v - u) \leq \varpi(\|\tilde{x} \ominus_g \tilde{y}\|)(\tilde{h}(v) - \tilde{h}(u))$$

(c) Obviously, for each finite step fuzzy-valued function  $\tilde{x}(t)$ ,  $\tilde{f}(t, \tilde{x})$  is fuzzy Henstock integrable.

From Theorem 14, there exist  $\Delta^-, \Delta^+ > 0$  s.t. on  $[t_0 - \Delta^-, t_0 + \Delta^+]$  there is a BV-solution  $\tilde{x}$  of (2) with  $\tilde{x}(t_0) = \tilde{x}_0$ .

This indicates that the GFDEs extends the theory of fuzzy differential equations based on gH-derivatives.

## 5 Uniqueness of BV-solutions for GFDEs

Since Bede and Gal (2005) proposed the generalized derivative and the generalized solutions of the FDEs, the solutions has lost its uniqueness. However, it can be proved that under the given conditions in this paper, the GFDEs has uniqueness with respect to the BV-solution.

**Definition 11** Let  $\tilde{x} : [t_0, t_0 + \eta] \rightarrow \mathbb{R}_{\mathcal{F}}$  be the BV-solution of problem (2). If for any solution  $\tilde{y} : [t_0, t_0 + \sigma] \rightarrow \mathbb{R}_{\mathcal{F}}$  of (2) with  $\tilde{y}(t_0) = \tilde{x}(t_0)$ ,  $\exists \eta_1 > 0$  s.t.  $\tilde{x}(t) = \tilde{y}(t)$  for  $t \in [t_0, t_0 + \eta] \cap [t_0, t_0 + \sigma] \cap [t_0, t_0 + \eta_1]$ , then  $\tilde{x}$  is locally unique.

To obtain the uniqueness theorems of the BV-solutions for problem (2), an important integral inequality is given.

**Lemma 1** (Schwabik (1992)) Let  $\psi : [a, b] \rightarrow [0, \infty)$ ,  $\tilde{h} : [a, b] \rightarrow [0, \infty)$  be given where  $\psi$  is bounded and  $\tilde{h}$  is continuous from the left and nondecreasing. Suppose that the function  $\varpi : [0, \infty) \rightarrow \mathbb{R}$  is nondecreasing and continuous with  $\varpi(0) = 0$ .

For  $u > 0$  let us set

$$\Psi(u) = \int_{u_0}^u \frac{1}{\varpi(r)} dr$$

with some  $u_0 > 0$ .

The function  $\Psi : (0, \infty) \rightarrow \mathbb{R}$  is an increasing function with  $\Psi(u_0) = 0$  and  $\lim_{u \rightarrow 0+} \Psi(u) = \alpha \geq -\infty$ ,  $\lim_{u \rightarrow \infty} \Psi(u) = \beta \leq \infty$ .

Assume that the inequality

$$\psi(\xi) \leq k + \int_a^\xi \varpi(\psi(\tau)) d\bar{h}(\tau)$$

holds for  $\xi \in [a, b]$ , where  $k > 0$ .

If  $\Psi(k) + \bar{h}(b) - \bar{h}(a) < \beta$  then

$$\psi(\xi) \leq \Psi^{-1}(\Psi(k) + \bar{h}(\xi) - \bar{h}(a))$$

where  $\xi \in [a, b]$ ,  $\Psi^{-1} : (\alpha, \beta) \rightarrow \mathbb{R}$  is the inverse function to  $\Psi$ .

**Theorem 15** Assume  $\tilde{f} \in \mathcal{V}(G, \bar{h}, \varpi)$  and for  $\forall u > 0$

$$\lim_{v \rightarrow 0+} \int_v^u \frac{1}{\varpi(r)} dr = \infty.$$

Then every BV-solution  $\tilde{x}$  of (2) s.t.  $\tilde{x}(t_0) = \tilde{x}_0$  where  $(\tilde{x}, t_0) \in G$  is locally unique.

**Proof** Assume that  $\tilde{x}, \tilde{y} : [t_0, t_0 + \eta] \rightarrow \mathbb{R}_{\mathcal{F}}$  are BV-solutions of problem (2) s.t.  $\tilde{x}(t_0) = \tilde{y}(t_0) = \tilde{x}_0$  where  $\eta > 0$ . Due to  $\int_{t_0}^t [\tilde{f}(s, \tilde{x}(s)) \odot_g \tilde{f}(s, \tilde{y}(s))] ds$  and  $\int_{t_0}^t \varpi(|\tilde{x}(s) \odot_g \tilde{y}(s)|) d\bar{h}(s)$  exists, for  $\forall \varepsilon > 0$ ,  $\exists \delta_1 : [t_0, t] \rightarrow (0, \infty)$  s.t. for  $\xi_i \in [u_{i-1}, u_i] \subset (\xi_i - \delta_1(\xi_i), \xi_i + \delta_1(\xi_i))$ ,

$$\begin{aligned} & \left\| \int_{t_0}^t [\tilde{f}(s, \tilde{x}(s)) \odot_g \tilde{f}(s, \tilde{y}(s))] ds \right\| \\ & \leq \left\| \int_{t_0}^t [\tilde{f}(s, \tilde{x}(s)) \odot_g \tilde{f}(s, \tilde{y}(s))] ds \right\| \\ & \quad \odot_g \sum_{i=1}^m [\tilde{f}(\xi_i, \tilde{x}(\xi_i)) \odot_g \tilde{f}(\xi_i, \tilde{y}(\xi_i))](u_i - u_{i-1}) \\ & \quad + \left\| \sum_{i=1}^m [\tilde{f}(\xi_i, \tilde{x}(\xi_i)) \odot_g \tilde{f}(\xi_i, \tilde{y}(\xi_i))](u_i - u_{i-1}) \right\| \\ & < \frac{\varepsilon}{2} + \sum_{i=1}^m \varpi(|\tilde{x}(\xi_i) \odot_g \tilde{y}(\xi_i)|)(\bar{h}(u_i) - \bar{h}(u_{i-1})), \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=1}^m \varpi(|\tilde{x}(\xi_i) \odot_g \tilde{y}(\xi_i)|)(\bar{h}(u_i) - \bar{h}(u_{i-1})) \\ & \leq \left\| \int_{t_0}^t \varpi(|\tilde{x}(s) \odot_g \tilde{y}(s)|) d\bar{h}(s) \right\| \\ & \quad - \sum_{i=1}^m \varpi(|\tilde{x}(\xi_i) \odot_g \tilde{y}(\xi_i)|)(\bar{h}(u_i) - \bar{h}(u_{i-1})) \\ & \quad + \left\| \sum_{i=1}^m \varpi(|\tilde{x}(\xi_i) \odot_g \tilde{y}(\xi_i)|)(\bar{h}(u_i) - \bar{h}(u_{i-1})) \right\| \\ & < \frac{\varepsilon}{2} + \int_{t_0}^t \varpi(|\tilde{x}(s) \odot_g \tilde{y}(s)|) d\bar{h}(s). \end{aligned}$$

Therefore

$$\begin{aligned} ||\tilde{x}(t) \ominus_g \tilde{y}(t)|| &= ||\int_{t_0}^t [\tilde{f}(s, \tilde{x}(s)) \ominus_g \tilde{f}(s, \tilde{y}(s))] ds|| \\ &< \varepsilon + \int_{t_0}^t \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) d\tilde{h}(s). \end{aligned}$$

This obtain

$$\begin{aligned} ||\tilde{x}(t) \ominus_g \tilde{y}(t)|| &\leq \int_{t_0}^t \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) d\tilde{h}(s) \\ &= \int_{t_0}^{t_0+\delta} \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) d\tilde{h}(s) \\ &\quad + \int_{t_0+\delta}^t \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) d\tilde{h}(s), \end{aligned}$$

where  $0 < \delta < t - t_0$ . Besides,

$$\begin{aligned} &\int_{t_0}^{t_0+\delta} \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) d\tilde{h}(s) \\ &= \varpi(||\tilde{x}(t_0) \ominus_g \tilde{y}(t_0)||) [\tilde{h}(t_0^+) - \tilde{h}(t_0)] \\ &\quad + \lim_{t_1 \rightarrow t_0^+} \int_{t_1}^{t_0+\delta} \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) d\tilde{h}(s) \\ &\leq \sup_{s \in [t_0, t_0+\delta]} \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) [\tilde{h}(t_0 + \delta) - \tilde{h}(t_0)] \\ &= A(\delta), \end{aligned}$$

because  $\varpi(||\tilde{x}(t_0) \ominus_g \tilde{y}(t_0)||) = \varpi(0) = 0$ . So  $\lim_{\delta \rightarrow 0^+} A(\delta) = 0$ , since  $\tilde{h}(t_0^+)$  exists. Therefore

$$||\tilde{x}(t) \ominus_g \tilde{y}(t)|| \leq A(\delta) + \int_{t_0+\delta}^t \varpi(||\tilde{x}(s) \ominus_g \tilde{y}(s)||) d\tilde{h}(s)$$

for  $t \in [t_0 + \delta, t_0 + \eta]$ .

Let  $u_0 > 0$  and

$$\Psi(u) = \int_{u_0}^u \frac{1}{\varpi(r)} dr.$$

For  $t \in [t_0 + \delta, t_0 + \eta]$  by Theorem 15,

$$||\tilde{x}(t) \ominus_g \tilde{y}(t)|| \leq \Psi^{-1}(\Psi(A(\delta)) + \tilde{h}(t) - \tilde{h}(t_0 + \delta))$$

which means  $\Psi^{-1}(\Psi(A(\delta)) + \tilde{h}(t_0 + \eta) - \tilde{h}(t_0 + \delta)) < \beta$  where  $\beta = \lim_{u \rightarrow \infty} \Psi(u) \leq \infty$ .

Evidently, we have

$$\Psi(A(\delta)) + \tilde{h}(t_0 + \eta) - \tilde{h}(t_0 + \delta) < \Psi(A(\delta)) + \tilde{h}(t_0 + \eta) - \tilde{h}(t_0 +)$$

and because  $\lim_{\delta \rightarrow 0^+} A(\delta) = 0$  and  $\lim_{u \rightarrow 0^+} \Psi(u) = -\infty$  we obtain

$$\lim_{\delta \rightarrow 0^+} \Psi(A(\delta)) + \tilde{h}(t_0 + \eta) - \tilde{h}(t_0 +) = -\infty.$$

So, there is  $\delta_0 > 0$  s.t.

$$\Psi(A(\delta)) + \tilde{h}(t_0 + \eta) - \tilde{h}(t_0 +) < \beta$$

for  $\delta \in (0, \delta_0)$ . Applying  $\Psi$  to (3), then

$$\Psi(|\tilde{x}(t) \ominus_g \tilde{y}(t)|) \leq \Psi(A(\delta)) + \tilde{h}(t) - \tilde{h}(t_0 + \delta),$$

and this means

$$\Psi(|\tilde{x}(t) \ominus_g \tilde{y}(t)|) - \Psi(A(\delta)) \leq \tilde{h}(t) - \tilde{h}(t_0 + \delta) \leq \tilde{h}(t) - \tilde{h}(t_0+).$$

Therefore,

$$\int_{A(\delta)}^{|\tilde{x}(t) \ominus_g \tilde{y}(t)|} \frac{1}{\varpi(r)} dr \leq \tilde{h}(t_0 + \eta) - \tilde{h}(t_0+)$$

for  $t \in [t_0 + \delta, t_0 + \eta]$  and  $\delta \in (0, \delta_0)$ .

Assume for  $t^* \in (t_0, t_0 + \eta]$   $|\tilde{x}(t^*) \ominus_g \tilde{y}(t^*)| = k > 0$ . Then

$$\int_{A(\delta)}^k \frac{1}{\varpi(r)} dr \leq \tilde{h}(t_0 + \eta) - \tilde{h}(t_0+) < \infty$$

for every  $\delta \in (0, \delta_0)$  s.t.  $\delta < t^* - t_0$ . Now it is possible to take  $\delta \rightarrow 0+$  for obtaining the inequality

$$\lim_{\delta \rightarrow 0+} \int_{A(\delta)}^k \frac{1}{\varpi(r)} dr \leq \tilde{h}(t_0 + \eta) - \tilde{h}(t_0+) < \infty,$$

which contradicts the known conditions. So, for  $t \in (t_0, t_0 + \eta]$ ,  $|\tilde{x}(t) \ominus_g \tilde{y}(t)| = 0$ .

Therefore,

$$|\tilde{x}(t) \ominus_g \tilde{y}(t)| \leq \int_{t_0}^t \varpi(|\tilde{x}(s) \ominus_g \tilde{y}(s)|) d\tilde{h}(s)$$

when  $\tilde{x}, \tilde{y} : [t_0, t_0 + \eta] \rightarrow \mathbb{R}_{\mathcal{F}}$  are (ii)-solutions of (2) s.t.  $\tilde{x}(t_0) = \tilde{y}(t_0) = \tilde{x}_0$  where  $\eta > 0$ .

For the fuzzy differential equation given in Example 3, we use the above conclusion to further prove the uniqueness of bounded variational solutions.  $\square$

**Example 6** Let  $\tilde{h}, \varpi$  and  $\tilde{f}$  are given in Example 3. Considering the uniqueness of BV-solution for problem (2) based on Theorem 15.

All we need to prove is

$$\lim_{v \rightarrow 0+} \int_v^u \frac{1}{\varpi(r)} dr = \infty.$$

Obviously,

$$\lim_{v \rightarrow 0+} \int_v^u \frac{c + b\|\tilde{A}\|}{t} dr = (c + b\|\tilde{A}\|) \cdot \lim_{v \rightarrow 0+} \int_v^u \frac{1}{t} dr = \infty.$$

That is the BV-solution given in Example 3 is unique.

For the fuzzy differential equation given in Example 4, we use the above conclusion to further prove the uniqueness of bounded variational solutions.

**Example 7** Similarly, let us show the uniqueness of BV-solution where  $\tilde{h}, \varpi$  and  $\tilde{f}$  come from Example 4. Since  $\varpi(t) = \frac{t}{c}$ , we have

$$\lim_{v \rightarrow 0+} \int_v^u \frac{c}{t} dr = c \cdot \lim_{v \rightarrow 0+} \int_v^u \frac{1}{t} dr = \infty.$$

That is the BV-solution given in Example 4 is unique.

## 6 Conclusions and future research

In order to obtain the connection between the fuzzy bounded variation function space and the GFDEs, we first discuss some properties of the fuzzy bounded variation function space and obtain that the solution of the GFDEs must be a fuzzy bounded variation function (Theorem 12), and the right-end function must be fuzzy Henstock integrable (Corollary 4).

Next, by using the embedded theory of fuzzy number space and Schader Tichonov's fixed point theorem, we prove the existence and uniqueness theorems of GFDEs, and extend the solutions of GFDEs from the fuzzy absolutely continuous function space to the fuzzy bounded variation function space.

The main drawback of this paper is that it's difficult to calculate from a numerical level. Due to the shortcomings of fuzzy Henstock integrals and  $g$  differences, which are difficult to calculate in practice, this paper only discusses them from a theoretical perspective and lacks numerical experiments.

Our results extend the results given in Schwabik (1992).

The next research are as follows.

- (i) Since the function of  $\Phi$ -bounded variation is the extension and development of the function of bounded variation (see Li and Wu 2003), we will study the fuzzy  $\Phi$ -bounded variation solutions of problem (2).
- (ii) The stability of the solutions for the GODEs is essential research in real analysis (see Schwabik 1992). Therefore, We will study the stability of the BV-solutions for problem (2).

The research on fuzzy differential equations still attracts the attention of many scholars. Some paper consider a wider range of fuzzy differential equations, such as the heat equation with fuzzy boundary initial values (see, Wasques 2024); Some paper consider fuzzy differential equations defined by different algebraic operations, such as  $\psi$ -cross operations (see, Laiate 2023). However, most paper are based on the  $gH$ -derivative of fuzzy-number-valued functions, instead of considering  $g$ -derivatives, it is impossible to consider a class of fuzzy differential equations (Example 1). This paper is based on  $g$ -derivative and considers the existence and uniqueness of fuzzy bounded variation solutions for GFDEs, expanding the space of solutions for fuzzy differential equations.

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**Data Availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## Declarations

**Conflict of interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

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