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On the asymptotic behaviour of solutions of the discrete generalized Emden–Fowler equation

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ABSTRACT

By the continuation theory for k -set contractions, this paper proves the existence of nontrivial decaying solutions to the sup-linear discrete generalized Emden–Fowler equation and the existence of asymptotically linear solutions to the sub-linear one.

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1. Introduction

The theory of nonlinear difference equations has been widely studied due to important applications of discrete models in many fields such as physics, computer science, economics, neural network, ecology, cybernetics, and so on. Oscillatory and non-oscillatory properties are studied in [1, 2, 10, 11]. Boundary value problems are investigated by [9, 17, 24]. Detailed classification and existence of such solutions to various generalized Emden–Fowler equations are proved in [1, 11–14, 16]. Using retract technique with Lyapunov approach, asymptotic representations of solutions of an Emden–Fowler equation are obtained by [3–6, 15]. This paper continues to study the interesting problem on asymptotic behaviour of solutions of the following discrete generalized Emden–Fowler equation:

$$\Delta^2 y_n = q_{n+1} |y_{n+1}|^\gamma \operatorname{sgn} y_{n+1}, \quad n \geq 1, \quad (1)$$

where Δ is the forward difference operator given by $\Delta y_n = y_{n+1} - y_n$ and $\Delta^2 y_n = \Delta(\Delta y_n) = \Delta y_{n+1} - \Delta y_n = y_{n+2} - 2y_{n+1} + y_n$. The class of Emden–Fowler equation appears in the study of polytropic and isothermal gas spheres, electron distribution in heavy atoms, etc. [12]. Our tool is the continuation theory for k -set contractions. Let $(X, |\cdot|_X)$, $(Y, |\cdot|_Y)$ be Banach spaces and $\Omega \subset X$ a bounded open set. The (Kuratowski) measure of

non-compactness for a bounded subset $S \subset X$ is defined by

$$\alpha_X(S) = \inf\{d > 0; S \text{ admits a finite cover by its subsets} \\ S = \bigcup_{i=1}^m S_i, \text{ diam}_X S_i \leq d\},$$

where $\text{diam}_X S_i = \sup_{x,y \in S_i} |x - y|_X$. A continuous and bounded map $\mathcal{N} : \overline{\Omega} \rightarrow Y$ is called k -set contractive if $\alpha_Y(\mathcal{N}S) \leq k\alpha_X(S)$ for all bounded sets $S \subset \overline{\Omega}$, where $k \geq 0$ is a constant.

Theorem 1.1 ([21]): Suppose that $\mathcal{L} : X \rightarrow Y$ is a Fredholm operator of index zero, $p \in Y$ is a fixed point, and $\mathcal{N} : \overline{\Omega} \rightarrow Y$ is k -set contractive with $k < \ell(\mathcal{L})$, where

$$\ell(\mathcal{L}) = \sup\{r \geq 0; r\alpha_X(S) \leq \alpha_Y(\mathcal{L}S) \text{ for bounded } S \subset X\}, \quad (2)$$

and $\Omega \subset X$ is bounded, open, and symmetric about $0 \in \Omega$. Assume further that

- (i) $\mathcal{L}x \neq r\mathcal{N}x + rp$ for $x \in \partial\Omega$, $r \in (0, 1)$.
- (ii) $[\mathcal{Q}\mathcal{N}x + \mathcal{Q}p, x] \cdot [\mathcal{Q}\mathcal{N}(-x) + \mathcal{Q}p, x] < 0$ for $x \in \ker \mathcal{L} \cap \partial\Omega$, where $[\cdot, \cdot]$ is some bilinear form on $Y \times X$ and $\mathcal{Q}$ is the projection of Y onto $\text{coker } \mathcal{L}$.

Then there exists $x \in \overline{\Omega}$ such that $\mathcal{L}x = \mathcal{N}x + p$.

Theorem 1.1 is a powerful tool in showing the existence of periodic solutions [8, 18–20, 23]. Note that (1) is a discrete analogue of the following generalized Emden–Fowler equation:

$$y'' = g(t)|y|^\nu \text{sgn } y, \quad t \geq 0. \quad (3)$$

Qi and Zong [22] extend Theorem 1.1 to the study of asymptotic behaviour of solutions of (3). We aim at this paper to extend asymptotic results in [22] to the difference Equation (1). However, the method for a prior estimates in [22] is not applicable here due to the lack of continuity of solutions. We shall work on a space consisting of truncated sequences to find the desired solutions and then extend them to the whole positive integers. Properties on asymptotic behaviour are preserved if only finitely many terms are neglected.

We introduce the idea and main results on asymptotic behaviour in Section 2. The existence of nontrivial decaying solutions and asymptotically linear solutions is proved in Sections 3 and 4, respectively.

2. The idea and main results

We shall prove existence results with Theorem 1.1 on appropriate sequence spaces to obtain asymptotic properties. Such spaces will be clear provided suitable transformations

of variables are taken. For convenience, let

$$f(y) = |y|^\gamma \operatorname{sgn} y = \begin{cases} y^\gamma, & y > 0, \\ 0, & y = 0, \\ -(-y)^\gamma, & y < 0. \end{cases} \quad (4)$$

If $\gamma \geq 1$, then $f \in C^1(\mathbb{R}, \mathbb{R})$. Putting

$$y_n = x_n + h_n, \quad p_n = q_{n+1}f(h_{n+1}) - \Delta^2 h_n, \quad n \geq 1, \quad (5)$$

(1) takes the form of

$$\Delta^2 x_n = q_{n+1}[f(x_{n+1} + h_{n+1}) - f(h_{n+1})] + p_n, \quad n \geq 1, \quad (6)$$

where h is a known suitable sequence to be determined later. Therefore, a solution x of (6) which decays faster than h gives rise to a solution y of (1) behaving asymptotically like h .

We illustrate this point with an example. Let $\mu, a > 0, b \neq 0$,

$$q_n = \frac{a}{n^\mu} + \frac{b}{n^{\mu+1}} + o\left(\frac{1}{n^{\mu+1}}\right). \quad (7)$$

Setting $h_n = cn^{-\nu}$, $c > 0$, a straightforward computation shows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{1}{(n-1)^\nu} - \frac{1}{n^\nu}}{\frac{1}{n^{\nu+1}}} &= \lim_{n \rightarrow \infty} \frac{n^\nu - (n-1)^\nu}{(n-1)^\nu} \cdot n = \lim_{n \rightarrow \infty} \frac{n^\nu - (n-1)^\nu}{n^\nu} \cdot n \\ &= \lim_{n \rightarrow \infty} \left[1 - \left(1 - \frac{1}{n}\right)^\nu \right] n = \nu, \\ \Delta h_{n-1} = h_n - h_{n-1} &= c \left[\frac{1}{n^\nu} - \frac{1}{(n-1)^\nu} \right] \sim \frac{-c\nu}{n^{\nu+1}} \quad (n \rightarrow \infty), \end{aligned}$$

and using

$$(1+s)^\gamma = 1 + \gamma s + \frac{\gamma(\gamma-1)}{2} s^2 + o(s^2) \quad (s \rightarrow 0)$$

to get

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^v} + \frac{1}{(n-1)^v} - \frac{2}{n^v}}{\frac{1}{n^{v+2}}} \\
 &= \lim_{n \rightarrow \infty} \frac{n^v(n+1)^v + n^v(n-1)^v - 2(n^2-1)^v}{(n-1)^v n^v (n+1)^v} n^{v+2} \\
 &= \lim_{n \rightarrow \infty} \frac{n^v(n+1)^v + n^v(n-1)^v - 2(n^2-1)^v}{n^{2v}} n^2 \\
 &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^v + \left(1 - \frac{1}{n}\right)^v - 2 \left(1 - \frac{1}{n^2}\right)^v \right] n^2 \\
 &= \lim_{n \rightarrow \infty} \left[1 + \frac{v}{n} + \frac{v(v-1)}{2n^2} + 1 - \frac{v}{n} + \frac{v(v-1)}{2n^2} - 2 \left(1 - \frac{v}{n^2}\right) + o\left(\frac{1}{n^2}\right) \right] n^2 \\
 &= v(v+1), \\
 & \Delta^n h_{n-1} = h_{n+1} - 2h_n + h_{n-1} \sim \frac{cv(v+1)}{n^{v+2}} \quad (n \rightarrow \infty).
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 q_n f(h_n) &= q_n h_n^\gamma = \left[\frac{a}{n^\mu} + \frac{b}{n^{\mu+1}} + o\left(\frac{1}{n^{\mu+1}}\right) \right] \frac{c^\gamma}{n^{\gamma v}} \\
 &= \frac{ac^\gamma}{n^{\mu+\gamma v}} + \frac{bc^\gamma}{n^{\mu+\gamma v+1}} + o\left(\frac{1}{n^{\mu+\gamma v+1}}\right) \quad (n \rightarrow \infty).
 \end{aligned}$$

Choose v, c so that

$$\mu + \gamma v = v + 2, \quad ac^\gamma = cv(v+1),$$

then

$$\begin{aligned}
 p_n &= \frac{bc^\gamma}{(n+1)^{\mu+\gamma v+1}} + o\left(\frac{1}{n^{\mu+\gamma v+1}}\right), \\
 \sup_{n \geq 1} |n^{\mu+\gamma v+1} p_n| &= \sup_{n \geq 1} |n^{v+3} p_n| < \infty.
 \end{aligned}$$

If x decays at a polynomial rate such that

$$\max \left\{ \sup_{n \geq 1} |n^{v+1} x_n|, \sup_{n \geq 1} |n^{v+2} \Delta x_n|, \sup_{n \geq 1} |n^{v+3} \Delta^2 x_n| \right\} < \infty,$$

by the intermediate theorem, there exists $\theta_n \in (0, 1)$ such that

$$|f(x_n + h_n) - f(h_n)| = \gamma |\theta_n x_n + h_n|^{\gamma-1} \cdot |x_n|, \quad n \geq 1.$$

Therefore,

$$\begin{aligned}
 & |n^{v+3} q_n [f(x_n + h_n) - f(h_n)]| \\
 &= \gamma |n^\mu q_n| \cdot n^{(\gamma-1)v} |\theta_n x_n + h_n|^{\gamma-1} \cdot |n^{v+1} x_n| \\
 &= \gamma |n^\mu q_n| \cdot |\theta_n n^v x_n + n^v h_n|^{\gamma-1} \cdot |n^{v+1} x_n|,
 \end{aligned}$$

which is bounded on $\mathbb{N}^+$. Thus both sides of (6) decay at the polynomial rate of order $\nu + 3$.

Given $\lambda > 0$ and $N \in \mathbb{N}^+$, define the quantities

$$\begin{aligned}\|x\|_{\lambda,N} &= \sup_{n \geq N} |n^\lambda x_n|, & \|x\|_{\lambda,N;1} &= \max\{\|x\|_{\lambda,N}, \|\Delta x\|_{\lambda+1,N}\}, \\ \|x\|_{\lambda,N;2} &= \max\{\|x\|_{\lambda,N}, \|\Delta x\|_{\lambda+1,N}, \|\Delta^2 x\|_{\lambda+2,N}\}\end{aligned}$$

and the spaces of sequences decaying at a polynomial rate

$$\begin{aligned}Y_{\lambda,N} &= \{x : [N, \infty) \cap \mathbb{N} \rightarrow \mathbb{R}; \|x\|_{\lambda,N} < \infty\}, \\ Y_{\lambda,N}^1 &= \{x \in Y_{\lambda,N}; \Delta x \in Y_{\lambda+1,N}\}, \\ Y_{\lambda,N}^2 &= \{x \in Y_{\lambda,N}; \Delta x \in Y_{\lambda+1,N}, \Delta^2 x \in Y_{\lambda+2,N}\}.\end{aligned}\tag{8}$$

For simplicity, we write Y_λ^ℓ and $\|\cdot\|_{\lambda;\ell}$ respectively instead of $Y_{\lambda,N}^\ell$ and $\|\cdot\|_{\lambda,N;\ell}$ with $N = 1$, where $\ell = 0, 1, 2$, and we have used the convention that $Y_{\lambda,N}^0 = Y_{\lambda,N}$ and $\|\cdot\|_{\lambda,N;0} = \|\cdot\|_{\lambda,N}$.

The first main result is formulated as follows. Its proof is put in Section 3.

Theorem 2.1: *Suppose that $\gamma > 1$, $\mu \in (0, 2)$, $\nu = \frac{2-\mu}{\gamma-1}$, $\lambda \geq \nu + 1$, and $q \in Y_\mu \setminus \{0\}$. Then for every $h \in Y_\nu^2 \setminus \{0\}$ such that $\gamma \|q\|_\mu \|h\|_\nu^{\gamma-1} < 1$ and $p = qf(h) - \Delta^2 h \in Y_{\lambda+2}$, (1) has at least one nontrivial solution $y \in Y_\nu^2$ with $y - h \in Y_\lambda^2$.*

As for the sub-linear case $\gamma < 1$, we consider quantities

$$\begin{aligned}\|x\|_\lambda &= \sup_{n \geq 1} |e^{\lambda n} x_n|, & \|x\|_{\lambda,1} &= \max\{\|x\|_\lambda, \|\Delta x\|_{\lambda+1}\}, \\ \|x\|_{\lambda,2} &= \max\{\|x\|_\lambda, \|\Delta x\|_{\lambda+1}, \|\Delta^2 x\|_{\lambda+2}\}\end{aligned}$$

and spaces

$$\begin{aligned}Z_\lambda &= \{x : \mathbb{N}^+ \rightarrow \mathbb{R}; \|x\|_\lambda < \infty\}, & Z_\lambda^1 &= \{x \in Z_\lambda; \Delta x \in Z_{\lambda+1}\}, \\ Z_\lambda^2 &= \{x \in Z_\lambda; \Delta x \in Z_{\lambda+1}, \Delta^2 x \in Z_{\lambda+2}\}.\end{aligned}\tag{9}$$

The second main result reads as follows. Its proof is put in Section 4.

Theorem 2.2: *Suppose that $\gamma \in (0, 1)$, $\lambda_0 > 0$, $q \in Z_{\lambda_0}$. Then for any $\lambda \in (0, \lambda_0)$, $a \neq 0$, $b \in \mathbb{R}$, (1) has at least one asymptotically linear solution y with $y_n - an - b \in Z_\lambda^2$.*

The transformation is slightly different from (5). Given $\lambda \in (0, \lambda_0)$, choose $y_0 > 0$ so that

$$k := \frac{\|q\|_\lambda}{y_0^{1-\gamma}} < \frac{\lambda^2}{e^{2\lambda}}. \quad (10)$$

By [22], there is an odd function $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ such that $\tilde{f}(y) = f(y)$ for large $|y|$, and

$$0 < \tilde{f}'(y) \leq \frac{1}{y_0^{1-\gamma}}, \quad y \in \mathbb{R}. \quad (11)$$

Then we consider the auxiliary difference equation

$$\Delta^2 y_n = q_{n+1} \tilde{f}(y_{n+1}), \quad n \geq 1. \quad (12)$$

For any $a \neq 0, b \in \mathbb{R}$, setting

$$y_n = x_n + an + b, \quad p_n = q_{n+1} \tilde{f}(an + a + b), \quad n \geq 1, \quad (13)$$

(12) is reduced to

$$\Delta^2 x_n = q_{n+1} [\tilde{f}(x_{n+1} + an + a + b) - \tilde{f}(an + a + b)] + p_n, \quad n \geq 1. \quad (14)$$

Consequently, a solution x of (14) which decays to zero at an exponential rate gives rise to an asymptotically linear solution y of (1).

3. The existence of decaying solutions

Following [22], we can prove basic properties for the space $(Y_{\lambda,N}^\ell, \|\cdot\|_{\lambda,N;\ell})$ on completeness, compactly embedding and continuity of measure for non-compactness with respect to λ . They are listed in the appendix for convenience of readers. Our first goal is to obtain the following two results.

Lemma 3.1: *The second-order difference operator*

$$\Delta^2 : Y_{\lambda,N}^2 \rightarrow Y_{\lambda+2,N}, \quad x \mapsto \Delta^2 x \quad (15)$$

is a Fredholm operator of index 0 such that

- (i) *The kernel $\ker \Delta^2$ equals $\{0\}$.*
- (ii) *The range $\text{ran } \Delta^2$ is $Y_{\lambda+2,N}$.*
- (iii) *The codimension $\text{codim } \Delta^2 = \dim Y_{\lambda+2,N} / \text{ran } \Delta^2$ is 0.*

Proof: Note that $\Delta^2 x = 0$ implies $x_n = C_1 n + C_2$ which equals the constant $\lim_{n \rightarrow \infty} x_n = 0$. Thus Δ^2 is injective.

Next we show that Δ^2 is surjective. For any $y \in Y_{\lambda+2,N}$,

$$|y_j| \leq \frac{\|y\|_{\lambda+2,N}}{j^{\lambda+2}}, \quad j \geq N.$$

Define

$$z_n = - \sum_{j=n}^{\infty} y_j, \quad n \geq N.$$

Then for $n \geq N$,

$$\Delta z_n = z_{n+1} - z_n = \sum_{j=n}^{\infty} y_j - \sum_{j=n+1}^{\infty} y_j = y_n,$$

i.e. $y = \Delta z$, and

$$\begin{aligned} |z_n| &\leq \|y\|_{\lambda+2,N} \sum_{j=n}^{\infty} \frac{1}{j^{\lambda+2}} \leq \|y\|_{\lambda+2,N} \sum_{j=n}^{\infty} \int_{j-1}^j \frac{dt}{t^{\lambda+2}} \\ &= \|y\|_{\lambda+2,N} \int_{n-1}^{\infty} \frac{dt}{t^{\lambda+2}} = \frac{\|y\|_{\lambda+2,N}}{-(\lambda+1)t^{\lambda+1}} \Big|_{n-1}^{\infty} \\ &= \frac{\|y\|_{\lambda+2,N}}{(\lambda+1)(n-1)^{\lambda+1}}, \\ |n^{\lambda+1}z_n| &\leq \left(\frac{n}{n-1}\right)^{\lambda+1} \frac{\|y\|_{\lambda+2,N}}{\lambda+1} \leq \left(\frac{N}{N-1}\right)^{\lambda+1} \frac{\|y\|_{\lambda+2,N}}{\lambda+1}. \end{aligned}$$

Hence $z \in Y_{\lambda+1}^1$. By the same method, the sequence

$$x_n = - \sum_{j=n}^{\infty} z_j, \quad n \geq N$$

fulfils $x \in Y_{\lambda}^2$ and $\Delta^2 x = y$ such that

$$\|x\|_{\lambda,N} \leq \left(\frac{N}{N-1}\right)^{\lambda} \frac{\|z\|_{\lambda+1,N}}{\lambda} \leq \left(\frac{N}{N-1}\right)^{2\lambda+1} \frac{\|y\|_{\lambda+2,N}}{\lambda(\lambda+1)}.$$

At last, (iii) is obvious. ■

Lemma 3.2: $\ell(\Delta^2) \geq 1$ for the Δ^2 given by (15).

Proof: In view of $\Delta^2 : Y_{\lambda,N}^2 \subset Y_{\beta,N}^2 \rightarrow Y_{\lambda+2,N} \subset Y_{\beta+2,N}$ for $\lambda > \beta > 0$, we shall show for each bounded $S \subset Y_{\lambda,N}^2$,

$$\alpha_{\beta,N;2}(S) \leq \alpha_{\beta+2,N}(\Delta^2 S),$$

which yields by Lemma A.6 the desired inequality

$$\alpha_{\lambda,N;2}(S) \leq \alpha_{\lambda+2,N}(\Delta^2 S).$$

Let $\zeta = \alpha_{\beta+2,N}(\Delta^2 S)$. By Lemma A.7, $\alpha_{\beta+1,N;1}(\Delta S) \leq \alpha_{\beta+2,N}(\Delta^2 S) = \zeta$. Lemma A.3 shows that $Y_{\lambda,N}^2$ is compactly embedded into $Y_{\beta,N}$. Hence $\alpha_{\beta,N}(S) = 0$. Given $\epsilon > 0$, there is a finite decomposition $S = \cup_{i=1}^m S_i$ with $\text{diam}_{\beta,N} S_i < \epsilon$.

From $\Delta S_i \subset \Delta S$ and $\Delta^2 S_i \subset \Delta^2 S$ it follows that $\alpha_{\beta+1,N;1}(\Delta S_i) \leq \zeta$ and $\alpha_{\beta+2,N}(\Delta^2 S_i) \leq \zeta$. There is a finite decomposition $\Delta S_i = \cup_{j=1}^{n_i} A_i^j$ such that $\text{diam}_{\beta+1,N;1} A_i^j \leq \zeta + \epsilon$. Setting $S_i^j = \{x \in S_i; \Delta x \in A_i^j\}$, it follows that

$$S = \cup_{i=1}^m \cup_{j=1}^{n_i} S_i^j, \quad \text{diam}_{\beta+1,N;1} \Delta S_i^j \leq \text{diam}_{\beta+1,N;1} A_i^j \leq \zeta + \epsilon.$$

For any $x, y \in S_i^j$,

$$\|x - y\|_{\beta,N} \leq \text{diam}_{\beta,N} S_i^j \leq \text{diam}_{\beta,N} S_i < \epsilon,$$

$$\|\Delta x - \Delta y\|_{\beta+1,N} \leq \|\Delta x - \Delta y\|_{\beta+1,N;1} \leq \text{diam}_{\beta+1,N;1} \Delta S_i^j \leq \zeta + \epsilon,$$

$$\|\Delta^2 x - \Delta^2 y\|_{\beta+2,N} \leq \|\Delta x - \Delta y\|_{\beta+1,N;1} \leq \zeta + \epsilon.$$

Thus $\text{diam}_{\beta,N;2} S_i^j \leq \zeta + \epsilon$ and $\alpha_{\beta,N;2}(S) \leq \alpha_{\beta+2,N}(\Delta^2 S)$. ■

Set $\phi(t) = \gamma \|q\|_\mu (t + \|h\|_\nu)^{\gamma-1}$ for $t \geq 0$. It is easy to check that ϕ is continuous and strictly increasing for $t \geq 0$. Conditions in Theorem 2.1 imply $0 < \phi(0) < 1$. Hence there is $\delta > 0$ such that $\phi(0) \leq \phi(t) < 1$ for $t \in [0, \delta]$. Since $\lambda > 1$, for any $R > \frac{\|p\|_{\lambda+2}}{1-\phi(\delta)}$, there exists an integer $N > 1$ such that

$$\left(\frac{N}{N-1}\right)^{\lambda+1} < \lambda, \quad \epsilon_1 := \frac{R}{(N+1)^{\lambda-\nu}} < \delta. \quad (16)$$

Then $\phi(\epsilon_1) < \phi(\delta) < 1$. Let $\Omega = B(0, R) \subset Y_{\lambda,N}^2$ be the open ball centred at 0 with radius R .

Define according to (6) the following nonlinear operator:

$$\mathcal{N} : x \in Y_{\lambda,N}^2 \mapsto (\mathcal{N}x)_n = q_{n+1}[f(x_{n+1} + h_{n+1}) - f(h_{n+1})], \quad n \geq N.$$

Then (6) with $n \geq N$ is equivalent to $\Delta^2 x = \mathcal{N}x + p$. The reason why we only consider (6) with $n \geq N$ is to simplify the estimation of the bound of $\mathcal{N}x$.

Lemma 3.3: $\mathcal{N}$ is a bounded and continuous map from $\overline{\Omega}$ to $Y_{\lambda+2,N}$ and is k -set contractive with $k = \phi(\delta) < 1$.

Proof: We first show that $\mathcal{N}$ maps $\overline{\Omega}$ into $Y_{\lambda+2,N}$. Let $x \in \overline{\Omega}$. By the intermediate value theorem, there is $\theta_{n+1} \in (0, 1)$ with

$$\begin{aligned} |n^{\lambda+2}(\mathcal{N}x)_n| &\leq (n+1)^{\lambda+2} |q_{n+1}[f(x_{n+1} + h_{n+1}) - f(h_{n+1})]| \\ &= |(n+1)^\mu q_{n+1}| \cdot \gamma |(n+1)^\nu (\theta_{n+1} x_{n+1} + h_{n+1})|^{\gamma-1} \cdot |(n+1)^\lambda x_{n+1}| \\ &\leq \gamma \|q\|_\mu \left[\frac{|(n+1)^\lambda x_{n+1}|}{(n+1)^{\lambda-\nu}} + \|h\|_\nu \right]^{\gamma-1} \cdot \|x\|_{\lambda,N} \\ &\leq \gamma \|q\|_\mu \left[\frac{R}{(N+1)^{\lambda-\nu}} + \|h\|_\nu \right]^{\gamma-1} R = \phi(\epsilon_1)R < \phi(\delta)R, \quad n \geq N. \end{aligned} \quad (17)$$

Hence $\mathcal{N}x \in Y_{\lambda+2,N}$ and $\mathcal{N}\overline{\Omega} \subset Y_{\lambda+2,N}$ is bounded by $\phi(\delta)R$. Similarly, for any $x, y \in \overline{\Omega}$ and $n \geq N$, there is $\theta_{n+1} \in (0, 1)$ with

$$\begin{aligned}
 & |n^{\lambda+2}(\mathcal{N}x - \mathcal{N}y)_n| \\
 & \leq |(n+1)^{\lambda+2}q_{n+1}[f(x_{n+1} + h_{n+1}) - f(y_{n+1} + h_{n+1})]| \\
 & = |(n+1)^\mu q_{n+1}| \cdot \gamma |(n+1)^\nu [\theta_{n+1}(x_{n+1} - y_{n+1}) + y_{n+1} + h_{n+1}]|^{\gamma-1} \\
 & \quad \cdot |(n+1)^\lambda (x_{n+1} - y_{n+1})| \\
 & \leq \gamma \|q\|_\mu \cdot \left| \frac{(n+1)^\lambda}{(n+1)^{\lambda-\nu}} [\theta_{n+1}x_{n+1} + (1-\theta_{n+1})y_{n+1}] + (n+1)^\nu h_{n+1} \right|^{\gamma-1} \|x - y\|_{\lambda,N} \\
 & \leq \gamma \|q\|_\mu \left[\frac{R}{(N+1)^{\lambda-\nu}} + \|h\|_\nu \right]^{\gamma-1} \|x - y\|_{\lambda,N} \leq \phi(\delta) \|x - y\|_{\lambda,N}, \tag{18}
 \end{aligned}$$

showing the uniform continuity of $\mathcal{N}$ on $\overline{\Omega}$.

At last, we show that $\mathcal{N}$ is k -set contractive. Let $S \subset \overline{\Omega}$ be bounded and $\eta = \alpha_{\lambda,N,2}(S)$. For any $\epsilon > 0$, there are finite decompositions such that

$$S = \cup_{i=1}^m S_i, \quad \text{diam}_{\lambda,N,2} S_i \leq \eta + \epsilon,$$

For $x, y \in S_i$, (18) yields

$$\|\mathcal{N}x - \mathcal{N}y\|_{\lambda+2,N} \leq \phi(\delta) \|x - y\|_{\lambda,N} \leq \phi(\delta) \|x - y\|_{\lambda,N;2} \leq \phi(\delta)(\eta + \epsilon).$$

Since ϵ can be arbitrarily small, $\alpha_{\lambda+2,N}(\mathcal{N}S) \leq \phi(\delta)\eta = k\alpha_{\lambda,N,2}(S)$. ■

Lemma 3.4: Suppose that all the conditions in Theorem 2.1 hold. If $\mathcal{L}x = r\mathcal{N}x + rp$ for some $x \in Y_{\lambda,N}^2$ with $\|x\|_{\lambda,N;2} \leq R$ and $r \in (0, 1)$, then $\|x\|_{\lambda,N;2} < R$.

Proof: Summing up the equation $\mathcal{L}x = r\mathcal{N}x + rp$ on $[N, n]$ and letting $n \rightarrow \infty$, we obtain

$$\begin{aligned}
 \sum_{j=N}^n \Delta^2 x_j &= \sum_{j=N}^n (\Delta x_{j+1} - \Delta x_j) = \Delta x_{n+1} - \Delta x_N \\
 &= r \sum_{j=N}^n [(\mathcal{N}x)_j + p_j], \\
 -\Delta x_N &= r \sum_{j=N}^{\infty} [(\mathcal{N}x)_j + p_j], \\
 \Delta x_{n+1} &= -r \sum_{j=n+1}^{\infty} [(\mathcal{N}x)_j + p_j].
 \end{aligned}$$

Therefore, for $n \geq N$,

$$\begin{aligned}
 |n^{\lambda+1} \Delta x_n| &\leq n^{\lambda+1} \sum_{j=n}^{\infty} j^{-\lambda-2} j^{\lambda+2} |(\mathcal{N}x)_j + p_j| \leq n^{\lambda+1} \sum_{j=n}^{\infty} j^{-\lambda-2} \|\mathcal{N}x + p\|_{\lambda+2,N} \\
 &\leq n^{\lambda+1} \|\mathcal{N}x + p\|_{\lambda+2,N} \sum_{j=n}^{\infty} \int_{j-1}^j \frac{dt}{t^{\lambda+2}} = n^{\lambda+1} \|\mathcal{N}x + p\|_{\lambda+2,N} \int_{n-1}^{\infty} \frac{dt}{t^{\lambda+2}} \\
 &= n^{\lambda+1} \|\mathcal{N}x + p\|_{\lambda+2,N} \left. \frac{t^{-\lambda-1}}{-\lambda-1} \right|_{n-1}^{\infty} \leq \frac{\|\mathcal{N}x + p\|_{\lambda+2,N}}{\lambda+1} \left(\frac{N}{N-1} \right)^{\lambda+1} \\
 &< [\phi(\delta)R + \|p\|_{\lambda+2}] \frac{\lambda}{\lambda+1} < \phi(\delta)R + \|p\|_{\lambda+2} < R,
 \end{aligned}$$

where (16) and (17) and the definition of R are used in the last inequality. In the same way,

$$\begin{aligned}
 \sum_{j=N}^n \Delta x_j &= \sum_{j=N}^n (x_{j+1} - x_j) = x_{n+1} - x_N \\
 -x_N &= \sum_{j=N}^{\infty} \Delta x_j, \quad x_{n+1} = - \sum_{j=n+1}^{\infty} \Delta x_j.
 \end{aligned}$$

Therefore, for $n \geq N$,

$$\begin{aligned}
 |n^{\lambda} x_n| &\leq n^{\lambda} \sum_{j=n}^{\infty} j^{-\lambda-1} j^{\lambda+1} |\Delta x_j| \leq n^{\lambda} \|\Delta x\|_{\lambda+1,N} \sum_{j=n}^{\infty} j^{-\lambda-1} \\
 &\leq n^{\lambda} \|\Delta x\|_{\lambda+1,N} \int_{n-1}^{\infty} \frac{dt}{t^{\lambda+1}} = n^{\lambda} \|\Delta x\|_{\lambda+1,N} \left. \frac{-1}{\lambda t^{\lambda}} \right|_{n-1}^{\infty} \\
 &\leq \frac{\|\Delta x\|_{\lambda+1,N}}{\lambda} \left(\frac{N}{N-1} \right)^{\lambda} \leq \|\Delta x\|_{\lambda+1,N} < R,
 \end{aligned}$$

and

$$\begin{aligned}
 |n^{\lambda+2} \Delta^2 x_n| &= rn^{\lambda+2} |(\mathcal{N}x)_n + p_n| \leq \|\mathcal{N}x + p\|_{\lambda+2,N} \\
 &< \phi(\delta)R + \|p\|_{\lambda+2} < R.
 \end{aligned}$$

Summing up, $\|x\|_{\lambda,N;2} < R$. ■

Proof of Theorem 2.1: (i) of Theorem 1.1 is true on $\overline{\Omega}$ by Lemma 3.4. The bilinear form $[\cdot, \cdot]$ and projection $\mathcal{Q}$ can be chosen arbitrarily due to the fact that $\ker \mathcal{L} = \{0\}$. Then by Theorem 1.1, Lemmas 3.1–3.4, (6) has at least one solution $x \in Y_{\lambda,N}^2$. Thus $y = x + h \in Y_{v,N}^2$ is a nontrivial solution of (1) on $[N, \infty) \cap \mathbb{N}$ with $\|y - h\|_{\lambda,N;2} < R$. By the recurrence relation

$$y_n = -y_{n+2} + 2y_{n+1} + f(y_{n+1}),$$

this solution extends naturally to a unique solution of (1) in Y_v^2 . ■

4. The existence of asymptotically linear solutions

We consider in this section (1) in the sub-linear case $\gamma < 1$. With a slight modification of the proofs, Lemmas 3.1 and 3.2 are true for the difference operator

$$\Delta^2 : Z_\lambda^2 \rightarrow Z_\lambda, \quad x \mapsto \Delta^2 x. \quad (19)$$

Lemma 4.1: Define according to (14) the nonlinear operator

$$\mathcal{N} : x \in Z_\lambda^2 \mapsto (\mathcal{N}x)_n = q_{n+1}[\tilde{f}(x_{n+1} + an + a + b) - \tilde{f}(an + a + b)], \quad n \geq 1.$$

Then $\mathcal{N} \in C(Z_\lambda^2, Z_\lambda)$ and is k -set contractive with $k \in (0, 1)$ defined by (10).

Proof: Let $x \in Z_\lambda^2$. By the intermediate value theorem and (11),

$$|\tilde{f}(u) - \tilde{f}(v)| = |\tilde{f}'(\xi)(u - v)| \leq \frac{|u - v|}{y_0^{1-\gamma}} \quad (20)$$

for all $u, v \in \mathbb{R}$. Consequently,

$$\begin{aligned} |e^{\lambda n}(\mathcal{N}x)_n| &\leq \frac{e^{\lambda(n+1)}}{y_0^{1-\gamma}} |q_{n+1}x_{n+1}| = \frac{e^{-\lambda(n+1)}}{y_0^{1-\gamma}} |e^{\lambda(n+1)}q_{n+1}e^{\lambda(n+1)}x_{n+1}| \\ &\leq \frac{e^{-\lambda(n+1)}}{y_0^{1-\gamma}} \|q\|_\lambda \cdot \|x\|_{\lambda,2} = ke^{-\lambda(n+1)} \|x\|_{\lambda,2}, \quad n \geq 1. \end{aligned}$$

Hence $\mathcal{N}x \in Z_\lambda$ and

$$\|\mathcal{N}x\|_\lambda \leq k\|x\|_{\lambda,2}. \quad (21)$$

Moreover, from (20) and

$$(\mathcal{N}x - \mathcal{N}y)_n = q_{n+1}[\tilde{f}(x_{n+1} + an + a + b) - \tilde{f}(y_{n+1} + an + a + b)]$$

it follows that

$$\|\mathcal{N}x - \mathcal{N}y\|_\lambda \leq k\|x - y\|_{\lambda,2}, \quad (22)$$

showing the uniform continuity of $\mathcal{N}$ on Z_λ^2 .

At last, we show that $\mathcal{N}$ is k -set contractive. Let $S \subset Z_\lambda^2$ be bounded and $\eta = \alpha_{\lambda,2}(S)$. For any $\epsilon > 0$, there is a finite decomposition $S = \cup_{i=1}^m S_i$ with $\text{diam}_{\lambda,2} S_i \leq \eta + \epsilon$. By (22),

$$\text{diam}_\lambda \mathcal{N}S_i \leq k \text{diam}_{\lambda,2} S_i \leq k(\eta + \epsilon).$$

Since ϵ can be arbitrarily small, $\alpha_\lambda(\mathcal{N}S) \leq k\eta = k\alpha_{\lambda,2}(S)$. ■

Lemma 4.2: If $\mathcal{L}x = r\mathcal{N}x + rp$ for some $x \in Z_\lambda^2$ and $r \in (0, 1)$, then $\|x\|_{\lambda,2} \leq \frac{\|p\|_\lambda}{\lambda^2 e^{-2\lambda} - k}$, where p is given by (13).

Proof: By (13),

$$\begin{aligned} |e^{\lambda n} p_n| &\leq |e^{\lambda(n+1)} q_{n+1} \tilde{f}(an + a + b)| \\ &= |e^{\lambda_0(n+1)} q_{n+1}| \cdot e^{(\lambda - \lambda_0)(n+1)} |an + a + b|^\gamma \\ &\leq \|q\|_{\lambda_0} e^{(\lambda - \lambda_0)(n+1)} |an + a + b|^\gamma \end{aligned}$$

for large n . Hence $p \in Z_\lambda$. From $\Delta^2 x_j = r[(\mathcal{N}x)_j + p_j]$ it follows that

$$\begin{aligned} \sum_{j=1}^n \Delta^2 x_j &= \sum_{j=1}^n (\Delta x_{j+1} - \Delta x_j) = \Delta x_{n+1} - \Delta x_1 = r \sum_{j=1}^n [(\mathcal{N}x)_j + p_j], \\ -\Delta x_1 &= r \sum_{j=1}^{\infty} [(\mathcal{N}x)_j + p_j], \quad \Delta x_{n+1} = -r \sum_{j=n+1}^{\infty} [(\mathcal{N}x)_j + p_j]. \end{aligned}$$

Therefore,

$$\begin{aligned} |e^{\lambda n} \Delta x_n| &\leq e^{\lambda n} \sum_{j=n}^{\infty} |(\mathcal{N}x)_j + p_j| = e^{\lambda n} \sum_{j=n}^{\infty} e^{-\lambda j} e^{\lambda j} |(\mathcal{N}x)_j + p_j| \\ &\leq e^{\lambda n} \|\mathcal{N}x + p\|_\lambda \sum_{j=n}^{\infty} e^{-\lambda j} \leq e^{\lambda n} \|\mathcal{N}x + p\|_\lambda \sum_{j=n}^{\infty} \int_{j-1}^j e^{-\lambda t} dt \\ &= e^{\lambda n} \|\mathcal{N}x + p\|_\lambda \int_{n-1}^{\infty} e^{-\lambda t} dt = \frac{e^\lambda}{\lambda} \|\mathcal{N}x + p\|_\lambda \\ &\leq \frac{e^\lambda}{\lambda} (k\|x\|_{\lambda,2} + \|p\|_\lambda) =: C, \quad n \geq 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} |e^{\lambda n} x_n| &\leq e^{\lambda n} \sum_{j=n}^{\infty} |\Delta x_j| = e^{\lambda n} \sum_{j=n}^{\infty} e^{-\lambda j} e^{\lambda j} |\Delta x_j| \\ &\leq e^{\lambda n} \|\Delta x\|_\lambda \sum_{j=n}^{\infty} e^{-\lambda j} \leq \frac{e^\lambda}{\lambda} \|\Delta x\|_\lambda \leq \frac{Ce^\lambda}{\lambda}, \\ \|\Delta^2 x\|_\lambda &= r \|\mathcal{N}x + p\|_\lambda \leq k\|x\|_{\lambda,2} + \|p\|_\lambda \leq \frac{C\lambda}{e^\lambda}, \end{aligned}$$

Summing up,

$$\|x\|_{\lambda,2} \leq C \max \left\{ 1, \frac{e^\lambda}{\lambda}, \frac{\lambda}{e^\lambda} \right\} = \frac{Ce^\lambda}{\lambda} = \frac{e^{2\lambda}}{\lambda^2} (k\|x\|_{\lambda,2} + \|p\|_\lambda).$$

Therefore,

$$\|x\|_{\lambda,2} \leq \frac{\frac{e^{2\lambda}}{\lambda^2} \|p\|_\lambda}{1 - \frac{e^{2\lambda}}{\lambda^2} k} = \frac{\|p\|_\lambda}{\lambda^2 e^{-2\lambda} - k}.$$

■

Proof of Theorem 2.2: In view of $\ker \mathcal{L} = \{0\}$, the bilinear form $[\cdot, \cdot]$ and projection $\mathcal{Q}$ can be chosen arbitrarily. Let $R = \frac{\|p\|_\lambda}{\lambda^2 e^{-2\lambda} - k} + 1$. The existence of a nonzero exponentially decaying solution $x \in B(0, R) \subset Z_\lambda^2$ of (14) follows from Theorem 1.1, Lemmas 4.1 and 4.2. Thus $y_n = x_n + an + b$ is an asymptotically linear solution of (12). Clearly, $|y_n|$ is large for large n . Hence y is a solution of (1) for large n , and it gives rise to a solution of (1) on $\mathbb{N}^+$ satisfying $y_n - an - b \in Z_\lambda^2$. ■

5. Final remarks

This paper proves the existence of nontrivial decaying solutions to the sup-linear discrete generalized Emden–Fowler equation and the existence of asymptotically linear solutions to the sub-linear one. The topic on asymptotic behaviours of solutions has been studied in [3, 5, 6, 11–16]. The coefficients of the Emden–Fowler equations in [12–14] are close to a power sequence, while the sequence q in (1) is bounded, and even decays at the infinity at either a polynomial or an exponential rate. The equation studied in [3, 5, 6, 15] is of the form

$$\Delta^2 u(n) \pm n^\alpha u^m(n) = 0. \quad (23)$$

As indicated by the example in Section 2, we need $b \neq 0$ in (7) so that in (5) the sequence p could decay faster than h . Hence our method is not adapted to (23). See also Example (5.1) below. On the other hand, previous works rely much on hard analysis about equations and inequalities, e.g. the retract technique with Lyapunov approach in [3, 5, 6, 15] and the detailed classification of solutions in [1, 11–14, 16]. Our approach is based on nonlinear functional analysis which could avoid some cumbersome calculations, and we mainly focus on properties of operators defined by the equation and related function spaces. Consequently, quite different types of conditions are required in our work. We only need to consider the cases of $\gamma > 1$ and $\gamma < 1$ separately. Conditions on decaying speed of functions and bounds of coefficients are imposed. As a consequence, we cannot get descriptions of asymptotic behaviours of solutions as detailed as those in previous works such as [1, 11–14, 16].

Example 5.1: Consider the following Emden–Fowler equation:

$$\Delta^2 y_n = \left[\frac{1}{(n+1)^\mu} + \frac{1}{(n+1)^{\mu+1}} \right] |y_{n+1}|^\gamma \operatorname{sgn} y_{n+1}, \quad n \geq 1, \quad (24)$$

where $\gamma > 1$ and $\mu \in (0, 2)$. Let

$$q_n = \frac{1}{n^\mu} + \frac{1}{n^{\mu+1}}, \quad h_n = \frac{c}{n^\nu}$$

with $\nu = \frac{2-\mu}{\gamma-1}$ and $c^{\gamma-1} = \nu(\nu+1)$. Then

$$p_n = q_{n+1}f(h_{n+1}) - \Delta^2 h_n = \frac{c^\gamma}{(n+1)^{\nu+3}},$$

which belongs to $Y_{\nu+3}$ as shown in Section 2. Moreover,

$$\|q\|_\mu = \sup_{n \geq 1} \left(1 + \frac{1}{n} \right) = 2, \quad \|h\|_\nu = \sup_{n \geq 1} c = c.$$

Therefore,

$$\gamma \|q\|_{\mu} \|h\|_{\nu}^{\gamma-1} = 2\gamma c^{\gamma-1} = 2\gamma \nu(\nu+1) < 1$$

holds for suitable ν . Thus for fixed $\gamma > 1$, if $\mu \in (0, 2)$ is close to 2, then all conditions in Theorem 2.1 are true. It follows that (24) has a solution $y \in Y_{\nu}^2$ with $\{y_n - \frac{c}{n^{\nu}}\} \in Y_{\nu+1}^2$.

Example 5.2: By Theorem 2.2, if $\gamma \in (0, 1)$, then the Emden–Fowler equation

$$\Delta^2 y_n = e^{-n-1} |y_{n+1}|^{\gamma} \operatorname{sgn} y_{n+1}, \quad n \geq 1$$

has an asymptotically linear solution y with $y_n - an - b \in Z_{\lambda}^2$ for any $\lambda \in (0, 1)$, $a \neq 0$ and $b \in \mathbb{R}$.

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Appendix. Sequences decaying at a polynomial rate

We prove in this section fundamental properties for the space $(Y_{\lambda,N}^\ell, \|\cdot\|_{\lambda,N;\ell})$. Note that in the norms a semicolon is used before the order ℓ of difference to distinguish it from N .

The following lemma is a special case of Proposition 7.4 in [7, p. 44] on the compactness criterion for sets in (c_0) which consists of real sequences decaying to zero at the infinity.

Lemma A.1: *A set $S \subset (c_0)$ is relatively compact if and only if S is bounded and equi-convergent, i.e. for each $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{Z}_+$ such that $|x_n| < \epsilon$ for all $n \geq N + 1$ and $x \in S$.*

Lemma A.2: *$(Y_{\lambda,N}^\ell, \|\cdot\|_{\lambda,N;\ell})$ is a Banach space for $\ell = 0, 1, 2$.*

Proof: We first prove that $Y_{\lambda,N}$ is complete. Let $\{x^{(m)}\} \subset Y_{\lambda,N}$ be a Cauchy sequence. Then $\{n^\lambda x_n^{(m)}\} \subset B([N, \infty) \cap \mathbb{N}, \mathbb{R})$ is a Cauchy sequence in the Banach space of bounded sequences with supremum norm. Let the limit be $y \in B([N, \infty) \cap \mathbb{N}, \mathbb{R})$ and define a sequence x by setting $x_n = n^{-\lambda} y_n$. Obviously, $x \in Y_{\lambda,N}$ and

$$\|x^{(m)} - x\|_{\lambda,N} = \sup_{n \geq N} |n^\lambda x_n^{(m)} - y_n| \rightarrow 0$$

as $m \rightarrow \infty$.

Next we show the completeness of $Y_{\lambda,N}^1$. Let $\{x^{(m)}\} \subset Y_{\lambda,N}^1$ be a Cauchy sequence. Then $\{x^{(m)}\} \subset Y_{\lambda,N}$ and $\{\Delta x^{(m)}\} \subset Y_{\lambda+1,N}$ are Cauchy sequences, whence converge to, say, $x \in Y_{\lambda,N}$ and $y \in Y_{\lambda+1,N}$, respectively. If $n \geq N$,

$$\lim_{m \rightarrow \infty} |n^\lambda [x_n^{(m)} - x_n]| = \lim_{m \rightarrow \infty} |n^{\lambda+1} [\Delta x_n^{(m)} - y_n]| = 0.$$

So

$$\lim_{m \rightarrow \infty} x_n^{(m)} = x_n, \quad y_n = \lim_{m \rightarrow \infty} [x_{n+1}^{(m)} - x_n^{(m)}] = x_{n+1} - x_n = \Delta x_n.$$

Therefore, $x \in Y_{\lambda,N}^1$ and $\lim_{m \rightarrow \infty} \|x_m - x\|_{\lambda,N;1} = 0$. The completeness of $Y_{\lambda,N}^2$ can be proved similarly. ■

Lemma A.3: $Y_{\lambda,N}^1$ and $Y_{\lambda,N}^2$ are compactly embedded into $Y_{\beta,N}$ for $\lambda > \beta > 0$.

Proof: A direct calculation shows that for $x \in Y_{\lambda,N}$,

$$|n^\beta x_n| = n^{\beta-\lambda} |n^\lambda x_n| \leq n^{\beta-\lambda} \|x\|_{\lambda,N}, \quad n \geq N.$$

Let $\{x^{(m)}\} \subset Y_{\lambda,N}^1$ be bounded and define sequences $\{y^{(m)}\}$ by

$$y_n^{(m)} = n^\beta x_n^{(m)}, \quad n \geq N$$

for $m \in \mathbb{Z}_+$. It follows that

$$|y_n^{(m)}| \leq n^{\beta-\lambda} \|x^{(m)}\|_{\lambda,N} \leq \|x^{(m)}\|_{\lambda,N}.$$

Therefore, $\{y^{(m)}\} \subset (c_0)$ is bounded and equi-convergent. By Lemma A.1, $\{y^{(m)}\}$ is relatively compact in (c_0) . Thus $\{x^{(m)}\}$ is relatively compact in $Y_{\beta,N}$.

At last, $Y_{\lambda,N}^2$ is compactly embedded into $Y_{\beta,N}$ since a set bounded in $Y_{\lambda,N}^2$ is also bounded in $Y_{\lambda,N}^1$. ■

Lemma A.4: Suppose that $x \in Y_{\lambda,N}^\ell$, $\ell = 0, 1, 2$, then

$$\lim_{\beta \rightarrow \lambda-} \|x\|_{\beta,N;\ell} = \|x\|_{\lambda,N;\ell}.$$

Proof: Let $x \in Y_{\lambda,N}$. Clearly, $\|x\|_{\beta,N} \leq \|x\|_{\lambda,N}$ for $0 < \beta \leq \lambda$. For any $\epsilon > 0$, there exists $n_\epsilon \geq N$ such that

$$\|x\|_{\lambda,N} - \epsilon < |n_\epsilon^\lambda x_{n_\epsilon}| \leq \|x\|_{\lambda,N}.$$

It follows that

$$\|x\|_{\beta,N} \geq |n_\epsilon^\beta x_{n_\epsilon}| = |n_\epsilon^\lambda x_{n_\epsilon}| n_\epsilon^{\beta-\lambda} > (\|x\|_{\lambda,N} - \epsilon) n_\epsilon^{\beta-\lambda}.$$

Consequently,

$$\|x\|_{\lambda,N} - \epsilon \leq \liminf_{\beta \rightarrow \lambda-} \|x\|_{\beta,N} \leq \limsup_{\beta \rightarrow \lambda-} \|x\|_{\beta,N} \leq \|x\|_{\lambda,N}.$$

Since $\epsilon > 0$ is arbitrary, the limit equality holds.

At last, the proven limit equality for $\ell = 0$ and $\max\{u, v\} = \frac{u+v+|u-v|}{2}$ yield the ones for $\ell = 1$ and 2. ■

Lemma A.5: Suppose that $S \subset Y_{\lambda,N}^\ell$ is bounded, $\ell = 0, 1, 2$, then

$$\lim_{\beta \rightarrow \lambda-} \text{diam}_{\beta,N;\ell} S = \text{diam}_{\lambda,N;\ell} S.$$

Proof: Denote $\Gamma = \text{diam}_{\lambda,N;\ell} S$. For any $\epsilon > 0$, there are $x, y \in S$ such that $\Gamma - \epsilon < \|x - y\|_{\lambda,N;\ell} \leq \Gamma$. By Lemma A.4,

$$\Gamma - \epsilon < \|x - y\|_{\beta,N;\ell} \leq \text{diam}_{\beta,N;\ell} S \leq \Gamma$$

for $0 < \lambda - \beta < 1$. Thus the conclusion holds. ■

Lemma A.6: Suppose that $S \subset Y_{\lambda,N}^\ell$ is bounded, $\ell = 0, 1, 2$, then

$$\lim_{\beta \rightarrow \lambda-} \alpha_{\beta,N;\ell}(S) = \alpha_{\lambda,N;\ell}(S).$$

Proof: For any finite decomposition $S = \cup_{i=1}^m S_i$, there is j with $\text{diam}_{\lambda,N;\ell} S_j \geq \alpha_{\lambda,N;\ell}(S)$. By Lemma A.5, $\text{diam}_{\beta,N;\ell} S_j \geq \alpha_{\lambda,N;\ell}(S) - \epsilon$ for $0 < \lambda - \beta < 1$. Hence $\alpha_{\lambda,N;\ell}(S) - \epsilon \leq \alpha_{\beta,N;\ell}(S) \leq \alpha_{\lambda,N;\ell}(S)$ and the conclusion holds. ■

Lemma A.7: $\ell(\Delta) \geq 1$ for

$$\Delta : Y_{\lambda+1,N}^1 \rightarrow Y_{\lambda+2,N}, \quad x \mapsto \Delta x.$$

Proof: Clearly, $\Delta : Y_{\lambda+1,N}^1 \subset Y_{\beta+1,N}^1 \rightarrow Y_{\lambda+2,N} \subset Y_{\beta+2,N}$ for $\lambda > \beta > 0$. We shall show for each bounded $S \subset Y_{\lambda+1,N}^1$,

$$\alpha_{\beta+1,N;1}(S) \leq \alpha_{\beta+2,N}(\Delta S),$$

which yields by Lemma A.6 the desired inequality

$$\alpha_{\lambda+1,N;1}(S) \leq \alpha_{\lambda+2,N}(\Delta S).$$

Let $\eta = \alpha_{\beta+2,N}(\Delta S)$. By Lemma A.3, $Y_{\lambda+1,N}^1$ is compactly embedded into $Y_{\beta+1,N}$. So $\alpha_{\beta+1,N}(S) = 0$. Given $\epsilon > 0$, there is a finite decomposition $S = \cup_{i=1}^m S_i$ with $\text{diam}_{\beta+1,N} S_i < \epsilon$.

Because $\Delta S_i \subset \Delta S$, $\alpha_{\beta+2,N}(\Delta S_i) \leq \eta$. There is a finite decomposition $\Delta S_i = \cup_{j=1}^{m_i} A_i^j$ with $\text{diam}_{\beta+2,N} A_i^j \leq \eta + \epsilon$. Setting $S_i^j = \{x \in S_i; \Delta x \in A_i^j\}$, it follows that

$$S = \cup_{i=1}^m \cup_{j=1}^{m_i} S_i^j, \quad \text{diam}_{\beta+2,N} \Delta S_i^j \leq \text{diam}_{\beta+2,N} A_i^j \leq \eta + \epsilon.$$

For any $x, y \in S_i^j$,

$$\|x - y\|_{\beta+1,N} \leq \text{diam}_{\beta+1,N} S_i^j \leq \text{diam}_{\beta+1,N} S_i < \epsilon,$$

$$\|\Delta x - \Delta y\|_{\beta+2,N} \leq \text{diam}_{\beta+2,N} \Delta S_i^j \leq \eta + \epsilon.$$

Thus $\text{diam}_{\beta+1,N;1} S_i^j \leq \eta + \epsilon$ and $\alpha_{\beta+1,N;1}(S) \leq \alpha_{\beta+2,N}(\Delta S)$. ■